The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 633–643

ASYMPTOTIC PROPERTIES OF NON-LINEAR LEAST SQUARES ESTIMATORS[1]

BY ROBERT I. JENNRICH

University of California, Los Angeles

1. Introduction. The purpose of this paper is to set forth conditions for the consistency and asymptotic normality of least squares estimators of non-linear parameters and to show that the Gauss-Newton iteration method of estimation is asymptotically numerically stable. Assume that

(a) *a sequence of real valued responses* y_t *has the structure*

$$y_t = f_t(\theta_0) + e_t, \qquad t = 1, 2, 3, \cdots,$$

where the f_t *are known continuous functions on a compact subset* Θ *of a Euclidean space and the* e_t *are independent identically distributed errors with zero mean and finite variance* $\sigma^2 > 0$. (*The values of* θ_0 *and* σ^2 *are unknown.*)

Any vector $\hat{\theta}$ in Θ which minimizes

$$Q_n(\theta) = n^{-1} \sum_{t=1}^{n} (y_t - f_t(\theta))^2 \tag{1}$$

will be called a least squares estimate of θ_0 based on the first n values of y_t. It is natural to ask if, under assumption (a), there always exists a least squares estimator, i.e., a measurable function of $y_1, \cdots, y_n$ whose values are least squares estimates. As will be shown in Lemma 2, the answer is "yes." We seek conditions which will guarantee the consistency and asymptotic normality of a sequence of least squares estimators.

The problem arises already in the case of elementary linear regression. Let $x_1, x_2, \cdots$ be a sequence of real numbers and let

$$y_t = \beta x_t + e_t, \qquad t = 1, \cdots, n.$$

Under the assumption of normally distributed errors the least squares estimator $\hat{\beta}$ of β is also normally distributed. Most of us believe that for large n the same result holds, at least approximately, even if the errors are not normally distributed. But this is not automatic. If for example $x_t = 1/t$ and the e_t have a uniform distribution, the estimator $\hat{\beta}$ fails to be consistent and fails to be asymptotically normally distributed. On the other hand if $x_t \equiv 1$, and the e_t have any distribution satisfying assumption (a), then $\hat{\beta}$ is consistent and asymptotically normally distributed.

Results for linear least squares estimation are given by Eicker [3] and by Grenander and Rosenblatt ([4], p. 244). These authors make weaker assumptions

Received 14 September 1966; revised 9 October 1968.

[1] This research was supported in part by the Air Force Office of Scientific Research Grant AF-AFOSR-4.59-66 and by the National Institutes of Health Grant FR-3.