The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 601–609

THE REMAINDER IN THE CENTRAL LIMIT THEOREM FOR MIXING STOCHASTIC PROCESSES

BY WALTER PHILIPP[1]

University of Illinois

1. Introduction. Let $\langle x_n, n = 1, 2, \cdots \rangle$ be a sequence of independent random variables centered at expectations and uniformly bounded by 1 almost surely. It follows from the Berry-Esseen theorem ([2], p. 288) that

$$P(s_N^{-1} \sum_{n \leq N} x_n < x) = \phi(x) + O(s_N^{-1}) \tag{1}$$

where we set $s_N^2 = \sum_{n \leq N} E(x_n^2)$ and $\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$. The constant implied by O is numerical. As is well known, in general, the order of magnitude of the estimate of the remainder cannot be improved.

In a paper to appear shortly [3] I investigated the central limit problem for mixing sequences of random variables; in particular, necessary and sufficient conditions were given for the central limit theorem to hold. In the present paper a modest attempt is made to estimate the remainder for such mixing stochastic processes. Unfortunately I was unable to show the central limit theorem in the above strong form (1) but could prove only that the order of magnitude of the remainder does not exceed $s_N^{-\frac{1}{2}} \log^3 s_N$ in case the random variables are uniformly bounded almost surely and satisfy a certain additional condition. Since in this direction nothing appears to be known and since the proof of the above statement turned out to be not quite so simple as I first anticipated I felt that it might be worthwhile to supply the details. Moreover, in a subsequent paper [5] the results are used to prove the law of the iterated logarithm for mixing stochastic processes.

2. Preliminaries and statement of the theorems. Let $\langle x_n, n = 1, 2, \cdots \rangle$ be a sequence of random variables centered at expectations with $\sup_n E(x_n^2) \leq 1$ and $s_N^2 = E(\sum_{n \leq N} x_n)^2 \to \infty$. Denote by $\mathfrak{M}_{ab}$ the σ-algebra generated by the events $\{x_n < \alpha\}$, $1 \leq a \leq n \leq b \leq \infty$. We shall be concerned with the following two mixing conditions:

$$|P(AB) - P(A)P(B)| \leq \psi(n)P(A)P(B) \tag{I*}$$

for all $A \varepsilon \mathfrak{M}_{1t}$, $B \varepsilon \mathfrak{M}_{t+n,\infty}$ with $\psi(n) \downarrow 0$ as $n \to \infty$

$$\sup_t \sup_{B \varepsilon \mathfrak{M}_{t+n,\infty}} |P(B \mid \mathfrak{M}_{1t}) - P(B)| \leq \varphi(n) \downarrow 0 \tag{II*}$$

with probability 1. (II*) is equivalent to (for a proof see [1])

(II*′) For any events $A \varepsilon \mathfrak{M}_{1t}$ and $B \varepsilon \mathfrak{M}_{t+n,\infty}$ we have

$$|P(AB) - P(A)P(B)| \leq \varphi(n)P(A).$$

Received 17 May 1968.

[1] Research supported in part by the National Science Foundation Grant GP 7537.