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ONE-SIDED TESTING PROBLEMS IN MULTIVARIATE ANALYSIS[1]

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1. Introduction. Suppose one obtains N independent observations from a p-dimensional normal distribution with mean μ and covariance matrix Σ. Under either of the assumptions (a) Σ is known or (b) $\Sigma = \sigma^2\Sigma_0$ with σ^2 unknown and Σ_0 known, the problem of testing $H:\mu = 0$ against the restricted alternative $K:\mu_i \geqq 0$, $i = 1, \cdots, p$, (with at least one inequality strict) has been studied extensively in the past ten years. Bartholomew, Chacko, Kudô, Nüesch, and Shorack have derived the likelihood ratio tests (LRT) and their null distributions and have studied their power functions. Computations of Bartholomew (1961a) and Nüesch (1964) show that the LRT's have substantially higher power than the usual χ^2 or F tests used for testing $\mu = 0$ against the unrestricted alternative $\mu \neq 0$. Abelson and Tukey have proposed simple tests based on the best linear contrast, and their idea has been extended by Schaafsma and Smid. Bartholomew's computations show that these tests are also substantially better than the usual tests, but neither the LRT nor the Abelson-Tukey test is uniformly more powerful than the other.

In this paper we study the above and related testing problems with restricted hypotheses or alternatives, *under the assumption that* Σ *is completely unknown.* Two procedures are considered: the LRT's and a family of tests based on the notions of Schaafsma and Smid. In Section 5 the LRT is derived for the general problem of testing $H:\mu \varepsilon \mathcal{P}_1$ vs. $K:\mu \varepsilon \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are positively homogeneous sets with $\mathcal{P}_1 \subset \mathcal{P}_2$, and it is shown that the power of the LRT approaches one as the distance from the hypothesis H becomes large. In Section 7 the exact null distribution of the LRT is obtained for the special case where $\mathcal{P}_1 = \{0\}$ and $\mathcal{P}_2 = \{\mu:\mu_i \geqq 0, i = 1, \cdots, p\}$. Since this distribution depends on the unknown matrix Σ, this result in itself cannot be used to obtain the level α rejection region of the LRT. In Section 6, however, sharp upper and lower bounds (as Σ varies) on the null distribution of the LRT statistic are derived for the more general case of a one-sided alternative, where $\mathcal{P}_1 = \{0\}$ and $\mathcal{P}_2 = \mathcal{C}$, a cone (see Section 2 for definitions). These bounds are independent of the particular cone $\mathcal{C}$, and the upper bound provides a simple formula for the level α cutoff point of the LRT. Similar results are given in Section 8 for the problem of testing a one-sided hypothesis against unrestricted alternatives, where $\mathcal{P}_1 = \mathcal{C}$, a cone, and $\mathcal{P}_2$ is the entire space. Cases where only a subset of the components of μ are tested are also discussed (Sections 6, 7, 8).

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