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ON THE EXPECTED VALUE OF A STOPPED STOCHASTIC SEQUENCE[1]

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1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space with an integrable stochastic sequence $(X_n, \mathfrak{F}_n, n \geqq 1)$ defined on it. By a stochastic sequence is meant that the $\mathfrak{F}_n$'s form an increasing sequence of σ-fields in $\mathfrak{F}$ and that each random variable X_n is $\mathfrak{F}_n$ measurable. A random variable t is called a stopping time if it is positive integer (possibly $+\infty$) valued and if the event $[t = n] \varepsilon \mathfrak{F}_n$ for each $n \geqq 1$. If $P[t < \infty] = 1$, then t is called a stopping rule. For any sequence of random variables $(Z_n, n \geqq 1)$ and a stopping time t, we define the expected value of the stopped sequence by $EZ_t = \int_{t<\infty} Z_t$ provided the integral exists (we permit $EZ_t = \infty$ or $EZ_t = -\infty$). We let Z^+ and Z^- denote respectively the positive and negative parts of a random variable Z, and $\mathfrak{B}(Z)$ denote the σ-field generated by a random variable Z (possibly vector valued). Given a collection of sets $\mathfrak{G}$, a set A in $\mathfrak{G}$ is said to be an atom of $\mathfrak{G}$ if $B \varepsilon \mathfrak{G}$ and $B \subset A$ implies that $P[B] = 0$ or $P[B] = P[A]$. $\mathfrak{G}$ is said to be non-atomic if it contains no atoms.

Recently, Dubins and Freedman [4] established that

$$(1) \quad (X_n, \mathfrak{F}_n, n \geqq 1) \text{ a martingale with } \sup E|X_n| = \infty \text{ implies that there exists a stopping time } t \text{ such that } E|X_t| = \infty.$$

In [2], this result is extended to the submartingale case. One might suspect that (1) would hold for some stopping rule or that the hypotheses of (1) would imply the existence of a stopping time t such that $EX_t^+ = \infty$. However simple examples exist in both cases ([4], p. 1505 and [1], p. 270 respectively) showing that such is not the case. Here we show that results in both of these directions are possible by certain modifications of the hypotheses of (1). The techniques developed in [2] and [4] were found to be useful here also. The natural setting for the results stated below is that of the general stochastic sequence as opposed to martingales in [4] and submartingales in [2]. As a corollary to the stated results for general stochastic sequences, it is shown in Corollary 2 that (1) and the corresponding result in [2] can be improved in the case where the X_n's are partial sums of independent random variables.

2. Results. Obviously, a necessary condition for the existence of a stopping rule t such that $E|X_t| = \infty$ is the existence of an unbounded stopping rule. The following lemma gives sufficient conditions for the existence of an unbounded stopping rule.

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