The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 427–433

MARTINGALE EXTENSIONS OF A THEOREM OF MARCINKIEWICZ AND ZYGMUND[1]

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1. Introduction. Suppose that $(d_n, n \geqq 1)$ is an orthonormal sequence of independent random variables and $(a_n, n \geqq 1)$ is a sequence of real numbers. Kac and Steinhaus [4] prove that if d_n^2 are uniformly integrable and $\sum a_n d_n$ converges a.s., then $\sum a_n^2 < \infty$. Marcinkiewicz and Zygmund [5] improve this result by replacing the uniform integrability of d_n^2 by $E|d_n| \geqq \delta > 0$ for every n. Recently Gundy [3] has extended the latter to martingales as follows: Let $(x_n = d_1 + \cdots + d_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale with $E(d_n^2 \mid \mathfrak{F}_{n-1}) = 1$ a.s. and $E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq \delta$ a.s. for some number $\delta \geqq 0$, and let $(\varphi_n, \mathfrak{F}_{n-1}, n \geqq 1)$ be a stochastic sequence, i.e., φ_n are $\mathfrak{F}_{n-1}$ measurable random variables. Then except on a null set, $\sum \varphi_n^2 < \infty$, $\sum \varphi_n^2 d_n^2 < \infty$ and $\sum \varphi_n d_n$ converges are equivalent.

In [6] (also in [7], p. 123), Zygmund proves the following summability result: Let d_n be independent, identically distributed random variables with $P[d_1 = \pm 1] = \frac{1}{2}$ and let $(a_{m,n}, m \geqq 1, n \geqq 1)$ be a double sequence of real numbers such that $\lim_{m\to\infty} a_{m,n} = a_n$, finite, for each n. If $\sum_{k=1}^{\infty} a_{m,k} d_k = T_m$ a.s. and $P[T_m \text{ converges}] > 0$, then $\sum a_n^2 < \infty$.

In Section 3, we shall give a new proof of Gundy's theorem and improve it slightly. In Section 4, Marcinkiewicz and Zygmund's theorem is extended to a summability result, which includes Zygmund's theorem as a special case.

2. Notation and lemmas. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $(\mathfrak{F}_n, n \geqq 1)$ be a sequence of σ-fields with $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$ and $(x_n, n \geqq 1)$ be a sequence of random variables. If $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ and x_n is $\mathfrak{F}_n$-measurable for each n, the sequence $(x_n, \mathfrak{F}_n, n \geqq 1)$ is said to be a stochastic sequence. We always put $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. For a set A, the indicator function of A is denoted by $I(A)$, and the integral $\int_A x\, dP$ is shortened as $\int_A x$. If $(x_n = d_1 + \cdots + d_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, the sequence $(d_n, \mathfrak{F}_n, n \geqq 1)$ is called a martingale difference sequence.

LEMMA 1. *Let $d \geqq 0$ be a random variable and $\mathcal{G} \subset \mathfrak{F}$ be a σ-field. Put $m = E(d \mid \mathcal{G})$ and $v = E^{\frac{1}{2}}(d^2 \mid \mathcal{G})$. If $\lambda \geqq 0$ is a $\mathcal{G}$-measurable random variable and $P[m < \infty] = 1$, then*

$$vP(d > \lambda v \mid \mathcal{G}) \geqq \lambda(m - 2\lambda v), \tag{1}$$

$$E(d^2 I[\lambda d < 1] \mid \mathcal{G}) \geqq (m - \lambda v^2)^2 \quad \text{on} \quad [m \geqq \lambda v^2]. \tag{2}$$

Received 23 April 1968.

[1] This research was supported by the National Science Foundation under Grant GP-06073.