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GENERALIZED MEANS AND ASSOCIATED FAMILIES OF DISTRIBUTIONS

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1. Introduction. The mean of a distribution and the sample mean are key concepts in the theory of statistics. The generalized means studied in this paper share important properties of the expectation, which is seen in this context as a distinguished member of a very large class.

There is an especially close relationship between the sample mean and the theory of estimation associated with one parameter exponential families of distributions. Each of the generalized means determines analogous one parameter families of distributions. Such families are introduced in Section 4. In sampling from such a distribution the maximum likelihood estimate of the generalized mean of the distribution is the generalized mean of the sample. Under appropriate regularity conditions it is a strongly consistent and asymptotically normal estimator.

In Section 2 the generalized means, called ϕ-means, and sample ϕ-means are defined and some of their properties examined. A minimizing property is proved, and they are shown to have the Cauchy mean value property. Also, an extension of Jensen's inequality is observed to be valid for r-means. Asymptotic properties of strong consistency and normality of sample ϕ-means are developed in Section 3. A study is made of conditions under which the sample ϕ-means are infinitely often, or all but finitely often, above or below the distribution ϕ-mean as sample size becomes infinite.

Guenther [6] has recently discussed briefly estimation of λ in sampling from the one parameter family of densities

$$f(x; \lambda) = \lambda x^{\lambda-1}, \qquad 0 < x < 1, \quad \lambda > 0. \tag{1.1}$$

The following observations relative to this example are, on the one hand, irrelevant from the point of view of the discussion in [6]; on the other hand, they do not at all indicate the scope of the present investigation. Nevertheless, they illustrate a partial motivation for considering means other than the usual sample mean. We note first that with changes of variable and of parameter, $Y = -\log X$, $\beta = 1/\lambda$, (1.1) becomes the ordinary exponential distribution. The sample mean is unbiased, sufficient, and efficient (in Cramér's finite sample sense); it is "the natural" estimate of β; the corresponding estimate of λ is then $n/(-\sum_i \log x_i)$. We observe further that (1.1) is a ϕ-family as defined in Section 4, with $\phi(x, \theta) = \log \theta - \log x$, $\theta(u) = \exp(-1/u)$; the ϕ-mean of

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