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A CHARACTERIZATION OF THE UPPER AND LOWER CLASSES IN TERMS OF CONVERGENCE RATES[1]

BY JAMES AVERY DAVIS

Sandia Corporation

For a given sequence of independent random variables $\{X_n\}$ a monotonic increasing positive sequence $\{\varphi_n\}$ is said to be in the upper class $\mathfrak{U}$ if

$$P[S_n > n^{\frac{1}{2}}\varphi_n \text{ infinitely often}] = 0,$$

where $S_n = \sum_{k=1}^{n} X_k$.

Otherwise $\{\varphi_n\}$ is in the lower class $\mathfrak{L}$ and the above probability is zero. In 1946, Feller [3] characterized these sequences as follows:

THEOREM (Feller). *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with*

$$EX_1 = 0, \qquad EX_1^2 = 1, \qquad \text{and} \quad \int_{|t|>y} t^2\, dF = O((\lg\lg y)^{-1}).$$

Then a monotonic increasing sequence $\{\varphi_n\}$ is in the upper class if and only if

$$\sum_{n=1}^{\infty} \varphi_n e^{-\varphi_n^2/2} n^{-1} < \infty.$$

The main result here is a characterization of the upper class in terms of a prescribed convergence rate for the partial sums of the random variables. This result represents an improvement of work previously done in this area. In [1] Baum and Katz show the following:

THEOREM [Baum and Katz]. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with*

(1) $EX_1 = 0$,

(2) $EX_1^2 = 1$,

(3) $EX_1^2(\lg|X_1|)^{1+\delta} < \infty$ *for some* $\delta > 0$.[2] *Then a monotonic increasing sequence $\{\varphi_n\}$ is in the upper class for $\{X_n\}$ if and only if*

$$\sum_{n=1}^{\infty} \varphi_n^2 n^{-1} P[S_n > n^{\frac{1}{2}}\varphi_n] < \infty.$$

In [2] the author shows that the same conclusion may be drawn if hypothesis (3) is weakened to $EX_1^2 \lg|X_1| \lg\lg|X_1| < \infty$. Here we obtain a similar conclusion under a moment condition slightly stronger than Feller's O-condition.

THEOREM. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$, and $EX_1^2 \lg\lg|X_1| < \infty$. Then a positive monotonic increasing sequence is in the upper class for $\{X_n\}$ if and only if*

$$\sum_{n=3}^{\infty} (\lg\lg n) n^{-1} P[S_n > n^{\frac{1}{2}}\varphi_n] < \infty$$

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[2] Throughout this paper $\lg X$ denotes the function $\lg X = \log_e X$ for $X > 1$ and 0 otherwise.