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A REMARK ON THE KOLMOGOROV-PETROVSKII CRITERION[1]

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Let $X(t, \omega)$ be any separable version of the standard Wiener process (Brownian motion) defined on a probability space $(\Omega, \mathcal{A}, P)$. Let ψ be any nonnegative function on $(0, \infty)$ such that $\lambda(t) = t^{-\frac{1}{2}}\psi(t)$ is monotone nondecreasing $(\uparrow)$. Define $T_\psi(\omega) = \sup\{t: X(t, \omega) \geqq \psi(t)\}$ and $\Lambda_\psi(\omega) = \lambda\{t: X(t, \omega) \geqq \psi(t)\}$ where λ is Lebesgue measure on $(0, \infty)$. The Kolmogorov-Petrovskii criterion (proved for coin tossing by Erdös) states that,

$$P[T_\psi < \infty] = 1 \tag{1}$$

if and only if,

$$Q(\psi) = \int_1^\infty t^{-3/2}\psi(t) \exp[-\tfrac{1}{2}\psi^2(t)t^{-1}]\, dt < \infty. \tag{2}$$

A beautiful treatment of these results is given in Strassen [3].

It is a trivial consequence of this criterion that if ψ is such that λ is $\uparrow$ and (2) holds then,

$$P[\Lambda_\psi < \infty] = 1. \tag{3}$$

The purpose of this note is to prove a partial converse of (3).

THEOREM. *Suppose ψ is such that λ is $\uparrow$ and*

$$\sup_{t \geqq 1} t^{-1}\psi(t) < \infty. \tag{4}$$

If $Q(\psi) = \infty$, then

$$P[\Lambda_\psi = \infty] = 1. \tag{5}$$

We begin with a lemma which is well known.

LEMMA. *Let $t_1 < t_2 < \cdots < t_n < \cdots$ where $t_n \uparrow \infty$ be a given sequence of numbers. Suppose $\mathcal{B}$ is the σ field generated by the variables $\{X(t_i, \cdot)\}$, $i \geqq 1$, and $\mathcal{B}_j$, $j \geqq 1$, is the σ field generated by the variables $\{X(t, \cdot)\}$, $t_{j-1} \leqq t < t_j$ where $t_0 = 0$. Then the σ fields $\mathcal{B}_1, \mathcal{B}_2, \cdots$, are conditionally independent given $\mathcal{B}$.*

PROOF. This is, of course, a general fact about Markov processes. It evidently suffices to check the independence of events $A_1, \cdots, A_r$ where $A_j \varepsilon \mathcal{B}_j$ is a cylinder set based on i_j of the variables $\{X(t, \cdot)\}$, $t_{j-1} \leqq t < t_j$, where r, i_j and the variables chosen are arbitrary. Let $\mathcal{B}^{(n)}$ be the σ field generated by $X(t_1, \cdot)$, $\cdots, X(t_n, \cdot)$. Since $\mathcal{B}_n \uparrow \mathcal{B}$ by the martingale convergence theorem it suffices to show that $A_1, \cdots, A_r$ are conditionally independent given $\mathcal{B}^{(n)}$ for all n sufficiently large. Therefore we need only check that if $X_1, \cdots, X_N$ is

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