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THE VARIANCE OF ONE-SIDED STOPPING RULES

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Let $x_1, x_2, \cdots$ be independent random variables with means $\mu_1, \mu_2, \cdots$ for which for some $0 < \mu < \infty$

$$n^{-1} \sum_1^n \mu_k \to \mu \qquad (n \to \infty). \tag{1}$$

Let $s_n = \sum_1^n x_k$, and for each $c > 0$ define

$$\begin{aligned} t = t(c) &= \text{first } n \geqq 1 \text{ such that } s_n > c \\ &= \infty \text{ if no such } n \text{ exists.} \end{aligned} \tag{2}$$

It is easily inferred from the results and methods of [5] that if

$$\sup_n n^{-1} \sum_1^n E(x_k - \mu_k)^- < \infty, \tag{3}$$

and if for each $\epsilon > 0$

$$\lim_{n\to\infty} n^{-1} \sum_1^n \int_{\{x_k - \mu_k > \epsilon n\}} (x_k - \mu_k) = 0, \tag{4}$$

then $Et < \infty$ for each $c > 0$ and $Et \sim c\mu^{-1} (c \to \infty)$. Under more restrictive conditions on the distributions of the x's an asymptotic expression for the variance of t may be obtained. To be specific, if the x's are identically distributed, non-negative, and if $\sigma^2 = Ex_1^2 - \mu^2 < \infty$, then it has been shown by Feller [2] in the lattice and Smith [6] in the non-lattice case that

$$\operatorname{Var} t \sim c\sigma^2\mu^{-3} \qquad (c \to \infty). \tag{5}$$

Recently, using combinational results of Spitzer [7], Heyde [4] has shown that (5) holds without the restriction to non-negative variables. The methods of Feller, Smith, and Heyde involve finding sufficiently detailed expansions of Et^2 and Et, from which (5) may be deduced. Smith and Heyde use Blackwell's Renewal Theorem.

In this note we generalize (5) to a large class of non-identically distributed x's. Our method involves Wald's lemma for squared sums [1] and the technique of Gundy and Siegmund [3] (see also [5]).

Theorem. *Let $x_1, x_2, \cdots$ be independent random variables with means μ_1, $\mu_2, \cdots$ such that for some $0 < \mu < \infty$*

$$\sum_1^n \mu_k - n\mu = o(n^{\frac{1}{2}}). \tag{6}$$

Let $\sigma_n^2 = Ex_n^2 - \mu_n^2$, $b_n^2 = \sum_1^n \sigma_k^2$ $(n = 1, 2, \cdots)$, and suppose that for some $0 < \sigma^2 < \infty$

$$b_n^2 \sim n\sigma^2. \tag{7}$$

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