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NOTES

AN L^p-CONVERGENCE THEOREM

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Recently Pyke and Root [4] strengthened a theorem of Marcinkiewicz proving that if f_n is a sequence of independent, identically distributed rv's with $\int |f_1|^p < \infty, 0 < p < 1$ and $\int f_1 = 0$ if $1 \leqq p < 2$ then $n^{-1/p}\{f_1 + \cdots + f_n\} \to 0$ a.s. and in L^p. The strengthening consisted in proving L^p-convergence. The purpose of this paper is to prove a similarly strengthened version of a theorem for martingales which is a generalization of the above-mentioned Marcinkiewicz's theorem. The a.s. convergence version of this theorem is in Loève [2], pp. 387. Our theorem contains that of Pyke and Root and is proved by using a tightened form of Minkowski's inequality due to Esseen and Von Bahr [1] which is stated as Lemma 1 and proved in a simple direct way.

LEMMA 1. *If $E(f_j \mid f_1 + \cdots + f_{j-1}) = 0$ (in particular if f_j is a martingale-difference sequence) for $2 \leqq j \leqq n$ and $f_j \in L^p$, $1 \leqq p \leqq 2$ then*

$$\int |f_1 + \cdots + f_n|^p \leqq \alpha \sum_{j=1}^n \int |f_j|^p$$

where $\alpha \leqq 2^{2-p} < 2$. (The actual value of α will be immaterial in the proof of the theorem.)

The cases $p = 1, 2$ being trivial, consider $1 < p < 2$. Here use the elementary inequality $|a + b|^p \leqq |a|^p + p|a|^{p-1} \cdot s(a)b + \alpha|b|^p$ for real numbers a, b ($s(a) =$ sign of a). This inequality follows easily from the fact that

$$\alpha = \sup_x \{|1 + x|^p - 1 - px\}/|x|^p$$

is finite. An elementary but tedious argument shows that $\alpha \leqq 2^{2-p} < 2$. Note also that $\alpha > 1$. Integrating the inequality we get $\int |f_1 + f_2|^p \leqq \int |f_1|^p + \alpha \int |f_2|^p$. Now apply induction.

THEOREM. *Let f_n, $n \geqq 1$, and f be measurable functions such that either $f \in L^p$, $0 < p < 2$, $p \neq 1$ and $P(|f_n| \geqq x) \leqq P(|f| \geqq x)$, $0 \leqq x < \infty$ or $f \in L^1$ and $P(|f_n| \geqq x | f_1 \cdots f_{n-1}) \leqq P(|f| \geqq x \mid f_1 \cdots f_{n-1})$ a.s. Then*

$$\lim_n n^{-1/p} \sum_{k=1}^n (f_k - \alpha_k) = 0 \quad \text{a.s.} \quad \textit{and in } L^p$$

where $\alpha_k = 0$ if $0 < p < 1$ and $\alpha_k = E(f_k \mid f_1 \cdots f_{k-1})$ if $1 \leqq p < 2$.

PROOF. The condition $P(|f_n| \geqq x) \leqq P(|f| \geqq x)$ with $f \in L^p$ implies that $f_n \in L^p$, $\sup_n \int |f_n|^p \leqq \int |f|^p$ and

(a) $\quad \sum_{n=1}^\infty P(A_n) < \infty \quad$ where $\quad A_n = \{|f_n| \geqq n^{1/p}\}$;

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