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EXPLICIT SOLUTIONS TO SOME PROBLEMS OF OPTIMAL STOPPING

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Suppose we are allowed to view successively as many terms as we please of a sequence $X_1, X_2, \cdots$ of independent random variables with common distribution F. We can decide to stop viewing at any time and if we decide to stop at time n, we receive the payoff $(X_1 + \cdots + X_n)/n$. How should we choose a stopping rule in order to maximize the expected payoff? This problem was introduced [5] in the context of inducing an illusory bias by selectively stopping an ESP experiment.

Based on their general theory of optimal stopping rules, Y. S. Chow and Herbert Robbins [9] succeeded in proving that an optimal rule exists when F is a two point distribution. They also proved the intuitively obvious but nontrivial fact that the unique minimal optimal rule is to stop at the first n at which $X_1 + \cdots + X_n \geqq \beta_n$, where $\beta_1, \beta_2, \cdots$ is a sequence of numbers, and gave a way to calculate β_n in principle. Aryeh Dvoretzky [14], and also H. Teicher and J. Wolfowitz [25] then proved that the same results hold for any F with finite second moment (the β's depend on F, of course). Dvoretzky also showed that if F has zero mean and unit variance then $0.32 < \beta_n/n^{\frac{1}{2}} < 4.06$ for n sufficiently large, and conjectured that $\lim \beta_n/n^{\frac{1}{2}}$ exists.

We prove the conjecture and find the value of the limit (which is independent of F as long as F has zero mean and unit variance) as the root $\alpha = 0.83992 \cdots$ of (1.3). The method is to use as an approximation the analogous continuous time problem, for which we can obtain the explicit optimal rule.

In the continuous time problem, also considered by Dvoretzky, the Wiener process $W(t), t \geqq 0$ is sampled continuously and stopping at time t gets the payoff $W(t)/(a + t)$. Dvoretzky pointed out that if $a > 0$ there exists an optimal stopping time. We show that there is a unique optimal stopping time and we find it explicitly: it is the first time τ that $W(\tau) = \alpha(a + \tau)^{\frac{1}{2}}$ (the same α as above). The expected payoff under the optimal rule is also given explicitly (Theorem 1). Except for the constant α, the parabolic form of the boundary determining τ is easily guessed by using the invariance of W under a change of scale. α can then be determined by using the "principle of smooth fit," due to Herman Chernoff and others for various special problems and treated carefully and in some generality by B. I. Grigelionis and A. N. Shiryaev [18]. However, to prove the optimality of τ rigorously we use a different approach, based on the fundamental Wald identity and on the work of Chow, Robbins, and Dvoretzky.

The continuous time problem discussed above is basically similar to the familiar Wald sequential probability ratio problem where, again on heuristic considerations of homogeneity, the optimal stopping rule is given in terms of a pair of

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