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WEAK APPROACHABILITY IN A TWO-PERSON GAME[1]

BY TIEN-FANG HOU[2]

University of California, Berkeley

1. Introduction. Let $M = \|m_{ij}\|$ be a 2×2 matrix whose elements m_{ij} are probability distributions on the Borel sets of a closed bounded convex subset X of Euclidean 2-space. We associate with M a game between two players, I and II, with the following infinite sequence of engagements: At the nth engagement, $n = 1, 2, \cdots$, player I selects a number p_n and player II selects a number q_n from the unit interval (each selection is made without either player knowing the choice of the other player), a point $Y_n \varepsilon X$ is selected according to the distribution

$$(1.1)\quad p_nq_nm_{11} + p_n(1 - q_n)m_{12} + (1 - p_n)q_nm_{21} + (1 - p_n)(1 - q_n)m_{22},$$

and then Y_n, p_n and q_n are announced to both players.

A strategy for player I is a function P defined on the set of all n tuples $(Y_1, p_1, q_1; \cdots; Y_n, p_n, q_n)$, $n = 1, 2, \cdots$, with values $P(Y_1, p_1, q_1; \cdots; Y_n, p_n, q_n) = p_{n+1}$ in the unit interval, and $p_1 = P(\varnothing)$, where $\varnothing$ is the empty sequence, is simply a point in the unit interval. A strategy Q for player II is similar: $Q(Y_1, p_1, q_1; \cdots; Y_n, p_n, q_n) = q_{n+1}$, $0 \leqq q_{n+1} \leqq 1$, and $q_1 = Q(\varnothing)$ is a point in the unit interval. For a given M, each pair P, Q of strategies determines a sequence of random variables $Y_1, Y_2, \cdots$ in X.

In this paper we investigate the controllability of the behavior of the random variable $\bar{Y}_N = \sum_1^N Y_n/N$ for each N, especially N large. For a given M and a set S in 2-space, can one of the players guarantee that $\bar{Y}_N$ is in or arbitrarily near S, with probability approaching 1 as N tends to infinite?

We paraphrase here the following definitions given by Blackwell [1]: For a given M, a set S in 2-space is said to be weakly approachable in M by $I(II)$ if for every $\nu > 0$ there is an N_0 such that, for every $N \geqq N_0$ there is a strategy P^* for I (Q^* for II) such that

$$(1.2)\qquad \text{Prob}\,\{\delta_N > \nu\} < \nu \qquad \text{for all} \quad Q(P),$$

where $\delta_N = \delta(\bar{Y}_N, S)$ denotes the distance of the point $\bar{Y}_N$ from S, and $Y_1, \cdots, Y_N$ are the variables determined by P^*, $Q(Q^*, P)$. The set S is weakly excludable in M by $I(II)$ if there exists a $\Delta > 0$ such that for every $\nu > 0$ there is an N_0 such that for every $N \geqq N_0$ there is a strategy P^* for I (Q^* for II)

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[2] Now at Bell Telephone Laboratories, Holmdel, New Jersey.