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ZEROES OF INFINITELY DIVISIBLE DENSITIES[1]

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In this note, we prove the following theorem, which grew out of a question posed by R. K. Getoor.

Theorem. *Suppose $\{p_t\,;t > 0\}$ is a convolution semigroup of probability density functions; i.e., for all s, $t > 0$ and for all $x \varepsilon (-\infty, \infty)$*

$$p_{s+t}(x) = \int_{-\infty}^{\infty} p_s(x - y)p_t(y)\,dy. \tag{1}$$

Suppose further that $p_t(x)$ is jointly continuous in $t \varepsilon (0, \infty)$ and $x \varepsilon (-\infty, \infty)$. Then $\{x:p_t(x) = 0\}$ is empty for all $t > 0$, or $\{x:p_t(x) = 0\}$ is a half-line $(-\infty, ct]$ or $[ct, \infty)$, where c is some constant.

Corollary. *If p is an infinitely divisible density function whose characteristic function φ has the property that all positive powers $|\varphi|^t$ are integrable, then the set of zeroes of p is either empty or a closed half-line.*

Proof of Corollary. If $|\varphi|^t$ is integrable, the probability measure μ_t corresponding to φ^t has a continuous density p_t, and the p_t form a convolution semigroup. Since

$$p_t(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixy}\varphi^t(y)\,dy,$$

and since

$$|\varphi(y)| \leqq 1,$$

the dominated convergence theorem shows that if $t_n \to t > 0$ and $x_n \to x$, then $p_{t_n}(x_n) \to p_t(x)$; in other words, $p_t(x)$ is jointly continuous in t and x.

Proof of Theorem. Let M be the Levy measure for the process

$$\{p_t(x)\,dx; t > 0\},$$

so that the characteristic function φ^t of p_t is given by

$$\varphi^t(u) = \exp\{itbu - t\delta^2u^2/2 + t\int [e^{iux} - 1 - iu \sin x]M(dx)\}.$$

Clearly, if $\delta^2 > 0$, then $p_t(x) > 0$ for all t and x, so it may be assumed that $\delta^2 = 0$. We then have, for every bounded continuous function f such that $f(x) = O(x^2)$ near 0,

$$\int f(x)t^{-1}p_t(x)\,dx \to' \int f(x)M(dx) \quad \text{as } t \to 0. \tag{2}$$

Since the distributions of the process are continuous, a well-known theorem of Hartman and Wintner [1] tells us that M must have infinite mass near 0. Without loss of generality, it may be assumed that M has infinite mass on the positive

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