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QUADRATIC FORMS AND IDEMPOTENT MATRICES WITH RANDOM ELEMENTS[1]

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1. Introduction. There have been a number of papers on the distribution of quadratic forms of normal variables [1], [2], [3], [4], [5]. The results are of particular importance in the theory of the general linear model, and idempotent matrices play a significant role in the distribution properties of quadratic forms for these models. In fact there are two basic results: let $\mathbf{y}$ be distributed as an $n \times 1$ normal random vector with mean $\boldsymbol{\mu}$ and positive definite covariance matrix $\mathbf{V}$. (1) $\mathbf{y}'\mathbf{A}\mathbf{y}$ is distributed as a non-central chi-square if and only if $\mathbf{AV}$ is idempotent; (2) $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'\mathbf{B}\mathbf{y}$ are independent if and only if $\mathbf{AVB} = \mathbf{0}$.

In the theorems in these papers mentioned above the matrices of the quadratic forms have constant elements. The purpose of this paper is to extend some of these theorems to the case where the elements of the matrices are random variables, and, of course, in these cases the function may no longer be a quadratic form in the observation vector $\mathbf{y}$.

2. Preliminary lemmas. In the theorems that we shall prove the basic variables will be assumed to have a multivariate normal distribution. Therefore, we shall state some results on this distribution.

DEFINITION 2.1. *Multivariate normal distribution of rank k.* Let $\mathbf{y}$ be an $n \times 1$ random vector with distribution function $F_y(\cdot)$ and characteristic function $\phi_y(\cdot)$. The vector $\mathbf{y}$ is defined to have a multivariate normal distribution of rank k if and only if the characteristic function of $\mathbf{y}$ is defined by

$$\phi_y(\mathbf{t}) = \exp\,(i\boldsymbol{\mu}'\mathbf{t} - \tfrac{1}{2}\mathbf{t}'\mathbf{V}\mathbf{t}); \quad \text{for all } \mathbf{t} \text{ in } n\text{-dimensional real space;}$$

where $\mathbf{V}$ is a non-negative (definite) $n \times n$ matrix of rank k and with constant elements, $\boldsymbol{\mu}$ is an $n \times 1$ vector of constant elements and $\boldsymbol{\mu}$ is in the column space of $\mathbf{V}$.

We shall also use the notation $\mathbf{y} \sim N(\boldsymbol{\mu}, \mathbf{V})$, $\mathbf{V}$ is $n \times n$ of rank k; to denote the distribution of $\mathbf{y}$.

We shall state a number of lemmas concerning the multivariate normal that we shall refer to later.

LEMMA 2.1. *Let $\mathbf{y}$ be defined in Definition 2.1. Then $\mathcal{E}(\mathbf{y}) = \boldsymbol{\mu}$; Cov $(\mathbf{y}) = \mathbf{V}$ where $\mathcal{E}(\cdot)$ denotes expectation and Cov $(\cdot)$ denotes a covariance matrix.*

LEMMA 2.2. *Let $\mathbf{y}$ be defined in Definition 2.1. Then there exists an $n \times k$ matrix $\mathbf{H}$ of rank k and a $k \times 1$ vector $\boldsymbol{\theta}$ such that $\mathbf{y} = \mathbf{H}(\mathbf{z} + \boldsymbol{\theta})$ where $\mathbf{z}$ is a $k \times 1$ vector*

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