The Annals of Mathematical Statistics
1969, Vol. 40, No. 4, 1401–1408

THE CANONICAL CORRELATION COEFFICIENTS OF BIVARIATE GAMMA DISTRIBUTIONS

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1. Introduction. Let $F(x, y)$ be a bivariate distribution function with marginal distribution functions $G(x)$ and $H(y)$. Lancaster [8] has studied the structure of bivariate distributions using orthogonal functions on the marginal distributions. Let $\{\zeta_i(x)\}$ and $\{\eta_j(y)\}$ be complete orthonormal sets of functions on $G(x)$ and $H(y)$ respectively such that $E(\zeta_i(x)\eta_j(y)) = \rho_j\delta_{ij}$, $1 \geqq \rho_1^2 \geqq \rho_2^2 \geqq \cdots$, where δ_{ij} is the Kronecker delta. $\{\zeta_i\}$, $\{\eta_j\}$ are called the canonical variables of (X, Y), and $\{\rho_i\}$ the canonical correlation coefficients of (X, Y). The sets $\{\zeta_i\}$, $\{\eta_j\}$ and $\{\rho_i\}$ determine the bivariate distribution function $F(x, y)$ uniquely given $G(x)$ and $H(y)$. $F(x, y)$ is said to be ϕ^2-bounded with respect to its marginal distributions if $\phi^2 + 1 = \int \{dF(x, y)/dG(x)dH(y)\}^2 \, dG(x)\, dH(y) < \infty$, or equivalently $\sum_{n=1}^{\infty} \rho_n^2 = \phi^2 < \infty$. ϕ^2-bounded distributions have a canonical expansion of the form $dF(x, y) = dG(x)\, dH(y)\{1 + \sum_{n=1}^{\infty} \rho_n \zeta_n^{(x)} \eta_n^{(y)}\}$, in mean square.

Sarmanov [10] has characterized the canonical correlation coefficients of ϕ^2-bounded distributions, whose marginal distributions are normal and whose canonical variables are the Hermite-Chebyshev polynomials. The series expansion of a bivariate normal frequency function in Hermite-Chebyshev polynomials,

$$(2\pi)^{-1}(1 - \rho^2)^{-\frac{1}{2}} \exp\{-(x^2 - 2\rho xy + y^2)/2(1 - \rho^2)\}$$
$$= (2\pi)^{-1} \exp\{-(x^2 + y^2)/2\}\{1 + \textstyle\sum_{n=1}^{\infty} \rho^n H_n(x) H_n(y)\}$$

is used in this characterization, $\{H_n(x)\}$ being orthonormal on $(2\pi)^{-\frac{1}{2}} \exp \cdot\{-x^2/2\}$. There is a similar expansion in the Laguerre polynomials of a bivariate gamma frequency function derived by Kibble [5]. A multivariate extension of this frequency function has been derived by Krishnamoorthy and Parthasarathy [7] and some properties of this multivariate case discussed by Krishnaiah and Rao [6].

In this note, the canonical correlation coefficients of bivariate gamma distributions, with canonical variables the Laguerre polynomials, are considered, making use of the frequency function derived by Kibble [5]. A class of these distributions which are ϕ^2-bounded is obtained, the general proof not depending on ϕ^2-boundedness.

The connection with Bochner's work [1] on stochastic processes is shown and thus a class of stochastic processes associated with the Laguerre polynomials is constructed.

Moran [9] has obtained a minimum bound for the ordinary correlation coeffi-

Received 7 December 1967; revised 3 March 1969.