The Annals of Mathematical Statistics
1969, Vol. 40, No. 4, 1236–1249

MOMENTS OF A STOPPING RULE RELATED TO THE CENTRAL LIMIT THEOREM[1]

By Bruce M. Brown

La Trobe University

0. Introduction. The main objective of this work has been to try and solve the following problem, which was introduced by Blackwell and Freedman and made the subject of further work by Chow and Teicher, Breiman, and Gundy and Siegmund.

Let $S_n = X_1 + \cdots + X_n$ be the sum of independent random variables $\{X_n\}$ with $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, with the stopping time $t_m(c) = \inf\{n : n \geqq m, |S_n| > cn^{\frac{1}{2}}\}$, $0 < c < \infty$, $m = 1, 2, \cdots$

(i) Is $P[t_m(c) < \infty] = 1$? (i.e. is $t_m(c)$ a stopping *rule*?)

(ii) For fixed $k (= 1, 2, \cdots)$, is $Et_m^k(c) < \infty$?

We have sought conditions on $\{X_n\}_1^\infty$, and values of m and c, such that (i) respectively (ii) hold. Before reaching a solution, which is given in Section 3, it was necessary to obtain some results in the related areas of martingales, stopping rules, and the Central Limit Theorem, so that Sections 1 and 2 are self-contained and possibly of independent interest.

1. Convergence of moments in the central limit theorem.

1.1. *Summary*. Let $\{X_n\}$ be independent rv's with $EX_n = 0$, $S_n = X_1 + \cdots + X_n$, and $s_n^2 = ES_n^2$ for $n = 1, 2, \cdots$, and $u_{i,j} = EX_j^i$, $v_{i,j} = E|X_j|^i$ for $i, j = 1, 2, \cdots$.
In (1.2) the Lindeberg condition of order ν, L_ν, is defined and in (1.3) it is shown that, when the central limit theorem holds, L_{2k} is necessary and sufficient for the convergence of $E(S_n/s_n)^{2k}$ to the $2k$th moment of a $N(0, 1)$ distribution.

Von Bahr, [11], has delved more fully into the question of convergence of moments in the central limit theorem, but his more involved results do not contain the present ones.

1.2. *Definitions*. A sequence of independent rv's $\{X_n\}$ with $EX_n = 0$ $S_n = X_1 + \cdots + X_n$, $s_n^2 = ES_n^2$, $n = 1, 2, \cdots$, is said to obey a Lindeberg condition of order $\nu \geqq 2$ (i.e. L_ν holds) if $s_n < \infty$, $n = 1, 2, \cdots$, and

$$(1) \qquad \sum_{j=1}^n \int_{[|X_j| \geqq \epsilon s_n]} |X_j|^\nu = o(s_n^\nu) \quad \text{as } n \to \infty \quad \text{for all } \epsilon > 0.$$

L_2 is the classical Lindeberg condition which is necessary and sufficient for the asymptotic normality of S_n/s_n, and $\max_{1 \leqq k \leqq n} EX_k^2 = o(s_n^2)$, as $n \to \infty$. Consider also

$$(2) \qquad \sum_{j=1}^n \int_{[|X_j| \geqq \epsilon s_j]} |X_j|^\nu = o(s_n^\nu) \quad \text{as } n \to \infty, \text{ all } \epsilon > 0;$$

and

$$(3) \qquad \sum_{j=1}^n E|X_j|^\nu = o(s_n^\nu) \quad \text{as } n \to \infty.$$

Received 22 May 1968; revised 4 February 1969.

[1] This work was supported by the National Science Foundation, Grant No. GP-06073.