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BAYES' METHOD FOR BOOKIES

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1. Introduction. Let Θ be a finite set, and for each $\theta \varepsilon \Theta$, let p_θ be a probability distribution on a finite set X. Consider three players: a master of ceremonies, a bookie, and a bettor. The master of ceremonies selects, at his pleasure, a θ belonging to Θ, and then an observation $x \varepsilon X$ at random, according to p_θ. He announces x to the bookie and the bettor. The bookie then posts odds on subsets of Θ, with the understanding that he must accept any combination of stakes the bettor might care to make. The bettor places his stakes. Finally, θ is revealed by the master of ceremonies and bookie and bettor settle up.

Before the game begins, how should the bookie plan to set the odds? One possibility is to choose a distribution on Θ, and when x is revealed, to calculate posterior odds by Bayes' rule. There is good reason for adopting this method. For any other procedure, there exists a system of bets with the following property: a bettor who places his stakes according to the system can expect to win money from the bookie, regardless of the θ chosen by the master of ceremonies. On the other hand, if the odds are calculated by Bayes' method, no such system exists. This is part of the content of Theorems 1 and 2 below. The two theorems extend a result of Bruno de Finetti (de Finetti, 1937, especially pages 6–8) which says (roughly) that someone who posts odds must do so on the basis of a finitely additive probability or else be certain to lose money to a clever bettor.

Section 2 of the paper treats the easier case where the odds are all finite and positive. The general case is developed in Section 3. Section 4, the final section, contains a theorem similar to the theorem of Section 2, but appropriate to situations involving prediction.

2. Finite, positive odds. Throughout this and the next section, the following assumptions will be in force. The sets Θ, X are finite and not empty. For each $\theta \varepsilon \Theta$, p_θ is a probability distribution on X. For each $x \varepsilon X$, the function $p.(x)$ is defined by the rule

$$p.(x): \theta \rightarrow p_\theta(x), \qquad \theta \varepsilon \Theta.$$

The complement of a subset A of Θ is written A^c, and A is said to be *proper* if neither A, A^c is empty. Everywhere A is a subset of Θ and x is a member of X.

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