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ON A CLASS OF NONPARAMETRIC TWO-SAMPLE TESTS FOR CIRCULAR DISTRIBUTIONS[1]

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0. Introduction. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent samples from circular distributions. A common problem consists in deciding whether the two samples have the same underlying distribution or not. In this paper we are primarily interested in non-parametric tests for the detection of rotation alternatives. Although there are "natural" isomorphisms between the circle and the interval $[0, 2\pi)$, the usual rank tests for detecting shift alternatives applied to observations in $[0, 2\pi)$ are not satisfactory, partly because they depend on an arbitrary cut-off point on the circle. Run tests can be adapted very easily to the circular two-sample problem, but their large-sample efficiency is zero for smooth families of distributions (Bahadur [1]). Kuiper [6] and Watson [12] suggested suitable modifications for the Kolmogorov-Smirnov and Cramér-von Mises tests. In this paper we use the invariance principle to derive a class of test statistics which is closely related to the class of rank tests for distributions on the real line.

1. Notation and assumptions. We define the unit circle as the set C of complex numbers of modulus 1. Then the natural isomorphism between $[0, 2\pi)$ and C is given by the mapping $x \to e^{ix}$. Under this isomorphism distributions and densities on C can be represented by cdf's and densities on $[0, 2\pi)$. For convenience we extend densities $f(\cdot)$ to all of R by the periodicity requirement $f(2k\pi + x) = f(x)$ $(k = \pm 1, \pm 2, \cdots)$.

In this paper we always assume that

(1.1) $$f(x) > 0 \quad \text{for almost all } x, \text{ and not a constant.}$$

(1.2) $$f'(x) \quad \text{exists and is continuous for all } x.$$

(1.3) $$\int_0^{2\pi} [f'(x)/f(x)]^2 \, dx = \inf(f) < \infty.$$

(1.4) $$m/(m+n) = \lambda_N \to \lambda \quad \text{with} \quad 0 < \lambda < 1, \quad \text{as} \quad N = m + n \to \infty.$$

2. Transformation group and invariant tests. As our class T of transformations of the sample space we take the set of all homeomorphisms of the circle onto itself, i.e., all bicontinuous, one-to-one mappings of C onto C. Any element $t \,\varepsilon\, T$ can be written in the form: $e^{ix} \to e^{i(c+t'(x))}$, $x \,\varepsilon\, [0, 2\pi]$, where $0 \leqq c < 2\pi$ and $t'(\cdot)$ is a bicontinuous (monotone) one-to-one mapping of $[0, 2\pi]$ onto

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