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ON DISTINGUISHING TRANSLATES OF MEASURES[1]

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1. Introduction. Let X denote a completely general real valued stochastic process on an arbitrary parameter set T. Let m be any real valued function on T. The well-known statistical problem of estimating a regression parameter consistently can be formulated as follows.

For any real number α, let P_α denote the probability measure that is induced by the stochastic process $x(t) + \alpha m(t)$ on the set S of all real valued functions on T. Let $\mathcal{A}$ denote the σ-field of subsets of S generated by coordinate functionals. (Thus a typical set in $\mathcal{A}$ is of the form $\{x \mid x(t_1) \varepsilon [0, \frac{1}{2}], x(t_2) \varepsilon [1, \infty)\}$ for $t_1, t_2 \varepsilon T$.) Let $\mathcal{A}_\alpha$ be the completion of $\mathcal{A}$ under P_α. Let $\mathcal{B}$ be the intersection of all the $\mathcal{A}_\alpha$'s. Then one may rigorously restate the question "Can one estimate α consistently" by asking whether there exists a functional f defined on S, measurable with respect to $\mathcal{B}$ and such that for all α, $P_\alpha[f = \alpha] = 1$.

In Section 2 of this paper we show how a criterion that Dudley [2] used to establish the singularity of the measures P_0 and P_1 can in fact be adapted to show the existence of such an f. To describe this criterion we need some more notation. Let S^0 denote the set of all "finitely defined" linear functionals on S. By this is meant that for any $f \varepsilon S^0$ there is a finite sequence $a_1, \cdots, a_n$ of real numbers and another finite sequence $t_1, \cdots, t_n$ of elements of T such that for all $x \varepsilon S, f(x) = \sum_1^n a_i x(t_i)$. Let $\mathfrak{J}$ be the pseudo-metric of convergence in P_0 measure. Then $(S^0, \mathfrak{J})$ is a pseudo-metric linear space. For any $m \varepsilon S$, $f \varepsilon S^0$ let $e_m(f) = f(m)$. The criterion of Dudley is just that e_m be a discontinuous linear functional on $(S^0, \mathfrak{J})$. In fact if this criterion is fulfilled then the functional f that we exhibit will be linear on the vector space S, hence the measures P_α are even "linearly singular."

In Section 3 we consider a certain subclass of processes with independent increments and show that all non trivial m give rise to discontinuous linear functionals on the pseudo-metric linear space just mentioned. In Section 4 we continue to treat processes with independent increments but no longer require that the functional f that distinguishes the measures P_α be linear. Dudley [2] under the hypotheses of Theorem 3 proves that the measures P_0 and P_1 are singular, and Gikhman and Skorokhod [3] under the hypotheses of Theorem 4 do the same. The theorems of this section extend the results of these authors in that the continuum of measures P_α are simultaneously distinguished.

2. Proof that the discontinuity criterion gives rise to a linear way of distinguishing the measures P_α.

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