The Annals of Mathematical Statistics
1969, Vol. 40, No. 5, 1721–1727

CHARACTERIZATIONS OF THE LINEAR EXPONENTIAL FAMILY IN A PARAMETER BY RECURRENCE RELATIONS FOR FUNCTIONS OF CUMULANTS[1]

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1. Introduction. The linear exponential family was characterized by a recurrence relation in cumulants by Patil [3] and by a recurrence relation in raw moments by Wani [4]. In this paper we present a general approach of characterizing the linear exponential family by a recurrence relation in functions of cumulants under the assumption that cumulants can be in turn expressed as functions of those occurring in the relation. Then, the characterization given by Wani [4] becomes a particular instance of ours since the raw moments can be expressed as functions of cumulants and vice versa.

2. The induced linear exponential family. A family $\mathcal{P}_e = \{P_\omega : \omega \in \Omega_\mu\}$ of probability distributions is said to be linear exponential in ω over a Euclidean sample space $(\mathfrak{X}, \beta)$ if

$$dP_\omega(x) = \{e^{\omega x}/f(\omega)\}\, d\mu(x) \tag{2.1}$$

where Ω_μ is assumed to be the natural parameter space with a nonvoid interior. It is understood by a natural parameter space that Ω_μ consists of all parameter points ω for which

$$f(\omega) = \int e^{\omega x}\, d\mu(x) \tag{2.2}$$

is positive and finite. Moreover, $f(\omega)$ is analytic in the interior of Ω_μ. If $\mathfrak{X}$ is p-dimensional, then we further assume that Ω_μ is a subset of a p-dimensional Euclidean space so that ωx can be interpreted as a scalar product of two vectors. We may call P_ω a linear exponential distribution in ω, but we bear in mind that P_ω may involve some other parameter in which it may not be linear exponential.

We observe that $f(\omega)$ is not unique; for any positive constant multiple of $f(\omega)$ gives rise to the same distribution P_ω. For example, given any interior point ξ of Ω_μ we may write (2.1) as

$$dP^*_{\theta,\xi}(x) = dP_\omega(x) = \{e^{\theta x}/m(\theta, \xi)\}\, dP_\xi(x) \tag{2.3}$$

where $\theta = \omega - \xi$ and $m(\theta, \xi) = f(\theta + \xi)/f(\xi)$. We can readily see that $m(\theta, \xi)$ is the moment generating function (mgf) of P_ξ with θ as its parameter. Thus we are led to define $P^*_{\theta,\xi}$ as an induced linear exponential distribution in θ of the distribution P_ξ. In fact, we can extend this definition to any distribution, not

Received 25 July 1968.

[1] This paper was prepared while the author was a fellow of the Summer Research Institute of the Canadian Mathematical Congress, 1968.

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