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HOMOGENEOUS GAUSS-MARKOV RANDOM FIELDS[1]

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1. Introduction. In this paper we consider real Gaussian random fields which are: (a) homogeneous with respect to the motions of an n-dimensional space of constant curvature, and (b) Markovian in the sense of Lévy [1]. The principal result of this paper is the characterization of such random fields in terms of their covariance functions. We recall that in one dimension a similar question has the very simple answer that the covariance function of a stationary Gauss-Markov process must be an exponential. The answer in the n-dimensional case is nearly as simple, and will be given in this paper.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a fixed probability space, and let $\{x(\omega, z), \omega \varepsilon \Omega, z \varepsilon V_n\}$ be a family of real Gaussian random variables with an n-dimensional parameter space V_n. We shall only consider three cases: (a) $V_n = R^n$, Euclidean space. (b) $V_n = S^n$, sphere. (c) $V_n = H^n$, hyperbolic space. Let $G(V_n)$ be the full group of motions in V_n which preserve distances. Suppose that for any finite set $A = \{z_i\} \subset V_n$, $\{x(\cdot, z_i), z_i \varepsilon A\}$ and $\{x(\cdot, gz_i), z_i \varepsilon A\}$ have the same distribution whenever $g \varepsilon G(V_n)$. Then we say $\{x(\cdot, z), z \varepsilon V_n\}$ is a homogeneous random field.

Markovian property in higher dimensions was introduced by Lévy [1] in connection with Brownian motion. Let ∂D be a smooth closed surface of dimension $n - 1$ in V_n, separating V_n into a bounded part D^-, and a possibly unbounded part D^+. A random field $\{x(z), z \varepsilon V_n\}$ is said to be Markovian of degree $\leqq p + 1$, if for any such ∂D every approximation $\hat{x}(z)$ to $x(z)$ in a neighborhood of ∂D which satisfies

$$|\hat{x}(z) - x(z)| = o(\delta^p) \qquad \delta = \text{distance } (z, \partial D)$$

also has the property that given $\hat{x}(\cdot)$, $x(z)$ and $x(z')$ are independent whenever $z \varepsilon D^-$ and $z' \varepsilon D^+$.

A random field is Markovian of degree p, if it is Markovian of degree $\leqq p$, but not $\leqq p - 1$. In this paper we are primarily concerned with Markovian fields of degree 1. For this special case it is more convenient to define the Markovian property by: given $\{x(z), z \varepsilon \partial D\}$, $x(z), z \varepsilon D^-$, and $x(z), z \varepsilon D^+$, are independent. If $x(z)$ has continuous sample functions, this definition clearly reduces to that of Lévy. This latter definition is more convenient when we have occasion later to consider the possibility of defining Markovian property for generalized random fields.

Since Gaussian distributions are uniquely determined by second order prop-

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