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CONDITIONS FOR OPTIMALITY AND VALIDITY OF SIMPLE LEAST SQUARES THEORY

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1. Notation and introduction. A matrix is denoted by a bold face letter such as $\mathbf{A}$, $\mathbf{X}$, $\mathbf{\Sigma}$ etc. For a matrix $\mathbf{X}$ of order $n \times m$

$R(\mathbf{X})$ represents the rank of $\mathbf{X}$.

$\mathfrak{M}(\mathbf{X})$ represents the linear space generated by the columns of $\mathbf{X}$.

$\mathbf{X}^-$ represents a g-inverse as defined by Rao (1962, 1966, 1967b).

$\mathbf{P}_X = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is the projection operator which projects arbitrary n-vectors onto $\mathfrak{M}(\mathbf{X})$.

$\mathbf{X}^\perp$ denotes a matrix of maximum rank such that $\mathbf{X}'\mathbf{X}^\perp = \mathbf{0}$.

$\mathbf{I}$ denotes an identity matrix. The order of $\mathbf{I}$ will usually not be explicitly mentioned but can always be determined from the context.

Consider the Gauss-Markoff model $(\mathbf{Y}, \mathbf{X\beta}, \mathbf{\Sigma})$ where $\mathbf{Y}$ is a vector of observations, $E(\mathbf{Y}) = \mathbf{X\beta}$ and $D(\mathbf{Y}) = \mathbf{\Sigma}$, $\mathbf{X}$ being a given matrix of order $n \times m$ and $\mathbf{\beta}$ a vector of unknown parameters. In the context of the discussion in the present paper, the model will be simply referred to as $(\mathbf{X}, \mathbf{\Sigma})$. The best linear unbiased estimator (BLUE) of an estimable parametric function $\mathbf{p}'\mathbf{\beta}$, where $\mathbf{p}$ is a vector, under the model $(\mathbf{X}, \mathbf{\Sigma})$ is a linear function $\mathbf{L}'\mathbf{Y}$ such that $E(\mathbf{L}'\mathbf{Y}) = \mathbf{p}'\mathbf{\beta}$ and $\mathbf{L}'\mathbf{\Sigma}\mathbf{L}$ is a minimum. It is well known that a BLUE under $(\mathbf{X}, \mathbf{\Sigma})$ can be obtained by the general method of least squares (see Rao, 1965, page 188 and Mitra and Rao, 1968).

The BLUE of $\mathbf{p}'\beta$ under $(\mathbf{X}_0, \mathbf{\Sigma}_0)$ is said to be $(\mathbf{X}, \mathbf{\Sigma})$ optimal if it is also the BLUE of $\mathbf{p}'\mathbf{\beta}$ under the model $(\mathbf{X}, \mathbf{\Sigma})$. The object of the present paper is to characterize the set of $(\mathbf{X}, \mathbf{\Sigma})$ such that for every estimable parametric function the BLUE under a given model $(\mathbf{X}_0, \sigma^2\mathbf{I})$ is $(\mathbf{X}, \mathbf{\Sigma})$-optimal. Further, the classes of $\mathbf{\Sigma}$ for which different statistical methods based on $(\mathbf{X}, \sigma^2\mathbf{I})$ remain valid have been obtained.

In previous papers Rao (1967a, 1968)[1] gave the necessary and sufficient contions for BLUE under $(\mathbf{X}, \mathbf{\Sigma}_0)$ to be $(\mathbf{X}, \mathbf{\Sigma})$-optimal, in which case the investigation was confined to the characterization of $\mathbf{\Sigma}$ only. Similar results, but not providing an exact representation of $\mathbf{\Sigma}$, were also obtained by Zyskind (1967), Watson (1967) and Kruskal (1968) in the special case of $\mathbf{\Sigma}_0 = \sigma^2\mathbf{I}$.

2. The main results.

LEMMA 2.1. *If for every estimable parametric function the* BLUE *under* $(\mathbf{X}_0, \sigma^2\mathbf{I})$ *is* $(\mathbf{X}, \sigma^2\mathbf{I})$*-optimal, it is necessary and sufficient that* $\mathbf{X}$ *is of the form*

$$\mathbf{X} = \mathbf{X}_0 + (\mathbf{I} - \mathbf{P}_{X_0})\mathbf{A}, \tag{2.1}$$

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[1] The results were first given in a lecture at the Fifth Berkeley Symposium in 1965.