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THE LOOSE SUBORDINATION OF DIFFERENTIAL PROCESSES TO BROWNIAN MOTION[1]

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1. Introduction. Our terminology is, in general, that of [6].

A *differential process* $\{X(t)/t \varepsilon [0, \infty)\}$ is a stochastic process with stationary, independent increments that is continuous in law and satisfies the initial condition $P[X(0) = 0] = 1$. We shall assume that our processes are separable and have sample paths that are almost surely right-continuous. A *random time* $\{Y(T)\}$ is a nonnegative differential process with sample paths that are almost surely nondecreasing.

Every differential process is an infinitely divisible process. That is the characteristic functions are of the form

$$f_{X(t)}(u) = \exp\{t\psi_X(u)\}, \tag{1.1}$$

where $\psi_X(u) = i\gamma_X u - \sigma_X^2 u^2/2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux/(1 + x^2))\, dM_X(x) \cdot \gamma_X$, σ_X^2, and M_X are the Lévy parameters uniquely associated with the infinitely divisible random variable $X(1)$. The Lévy spectral function M_X is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$, is asymptotically zero $(M_X(-\infty) = 0 = M_X(+\infty))$, and satisfies the integrability condition

$$\int_{-1}^{-0} + \int_{+0}^{+1} x^2\, dM_X(x) < \infty.$$

The Lévy spectral function for a random time vanishes on the negative half-axis and satisfies the stronger integrability condition

$$\int_{+0}^{+1} x\, dM_Y(x) < \infty.$$

Consequently, the characteristic functions for a random time can be written in the form

$$f_{Y(T)}(u) = \exp\{T(i\gamma_Y u + \int_0^{\infty} (e^{iux} - 1)\, dM_Y(x))\}, \tag{1.2}$$

where $\gamma_Y \geqq 0$ is the trend term of the random time.

A standard Brownian motion $\{W(t)\}$ is a separable differential process with sample paths that are almost surely continuous and such that $\mathfrak{L}(W(t)) = \mathfrak{N}(0, t)$. Any random time $\{Y(T)\}$ independent of the standard Brownian motion is *loosely subordinate* in the sense that there exists a random time (*loose sub-*

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