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ESTIMATION OF PARAMETERS IN A TRANSIENT MARKOV CHAIN ARISING IN A RELIABILITY GROWTH MODEL[1]

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1. Introduction. We consider the following reliability growth model. Initially a device has probability p of failure. We subject it to a sequence of trials, making no changes if there is no failure. If there is a failure on any trial, then changes are made in the device which cause the probability of failure on the next trial to be reduced by the factor β, where $0 < \beta < 1$. Thus if there have been k failures the probability of failure on the next trial is $p\beta^k$. Let y_i be the number of failures up to and including trial i, with $y_0 = 0$. Then y_i, $i \geqq 0$, is a Markov chain, which may be regarded as a random walk on the nonnegative integers with the transition probabilities

$$(1.1)\quad P(y_{i+1} = k+1 \mid y_i = k) = p\beta^k, \qquad P(y_{i+1} = k \mid y_i = k) = 1 - p\beta^k.$$

In this paper we prove that the likelihood equations for the Markov chain y_i, $i \geqq 0$ have solutions, $\hat{p}$ and $\hat{\beta}$, which converge in probability to the true parameter values, p_0 and β_0, and which are asymptotically jointly normally distributed.

The Markov chain (1.1) is clearly transient, and we may regard this work as an example of the theory of estimation in chains of that type. Billingsley [2] has developed a general theory of estimation in Markov processes but his results do not apply here since his basic assumption is that the Markov process possesses a unique stationary distribution.

An estimation problem for a sequence of independent but not identically distributed random variables ξ_k which is closely related to the estimation problem for the Markov chain (1.1) is arrived at by defining ξ_k, $k \geqq 0$, to be the number of times state k is occupied in the infinite sequence $y_0, y_1, y_2, \cdots$. Clearly, the random variables ξ_k so defined are independent and have the geometric distributions

$$(1.2)\qquad P(\xi_k = x) = p\beta^k(1 - p\beta^k)^{x-1}, \qquad x = 1, 2, \cdots.$$

The sequence ξ_k, $k \geqq 0$ provides an alternative description of the reliability growth model considered here, in that for any $k \geqq 1$ the partial sum $\xi_0 + \xi_1 + \cdots + \xi_{k-1}$ represents the first trial (i) for which the accumulated number of failures (y_i) equals k. In this paper we first consider the estimation problem for a sample ξ_k, $k = 0, 1, \cdots, n-1$ and prove that the likelihood

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