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SUFFICIENT CONDITIONS FOR A MIXTURE OF EXPONENTIALS TO BE A PROBABILITY DENSITY FUNCTION

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1. Introduction. We shall consider the function

$$f(x) = \sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i x} \qquad (x \geqq 0) \tag{1}$$

where the λ's are positive and $\sum_{i=1}^{k} p_i = 1$. Without loss of generality we may suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. If all of the p's are positive then it is obvious that (1) represents a probability density function. If some of the p's are negative, $f(x)$ could be negative for some values of x and so may not be a density function. Steutel, in [3], remarked that there appear to be no simple conditions for determining whether or not $f(x)$ is a density. It is the main purpose of this note to provide some simple sufficient conditions. These all have the form of inequalities involving linear functions of the p's; the principal results are given in Theorems 1 and 2 and their Corollaries.

Mixed exponential distributions with negative p's arise in several statistical contexts. One of the best known members of the family is the so-called Erlang distribution. This plays a central role in the derivation of our conditions. It arises as follows. Let $y_1, y_2, \cdots y_k$ be independently and exponentially distributed random variables with scale parameters $\lambda_1, \lambda_2, \cdots \lambda_k$ respectively, then $x = \sum_{i=1}^{k} y_i$ has an Erlang distribution with density given by (1) with

$$p_i = \prod_{j=1, j\neq i}^{k} (\lambda_j/(\lambda_j - \lambda_i)), \qquad (i = 1, 2, \cdots k).$$

(see [1], page 17). In this case the signs of the p's alternate.

The mixed exponential distribution has many attractive properties. The fact that its Laplace transform is a rational algebraic fraction offers many advantages in renewal theory and other branches of stochastic processes. Kingman in [2] has shown that one can approximate arbitrarily closely to any density on $(0, \infty)$ by a function of the form (1), (although this might require a very large value of k in any particular instance). The many advantages which this function offers are off-set to some extent by the difficulty of determining whether a given $f(x)$ is in fact a density function. Our results provide a partial answer to this problem.

A further property, which has a direct application to testing for positivity, is that a mixed exponential distribution, truncated on the left, is also mixed exponential in form. Thus if the point of truncation is $x = X$ and if $y = x - X$ then

$$f(y \mid y \geqq 0) = \sum_{i=1}^{k} p_i' \lambda_i e^{-\lambda_i y} \tag{2}$$

where $p_i' = p_i e^{-\lambda_i X} / \sum_{i=1}^{k} p_i e^{-\lambda_i X}$.

2. Two necessary conditions. By considering the two cases $x = 0$ and $x \to \infty$,

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