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THE RELATION OF THE EQUIVALENCE CONDITIONS FOR THE BROWNIAN MOTION TO THE EQUIVALENCE CONDITIONS FOR CERTAIN STATIONARY PROCESSES[1]

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Random processes are called equivalent if the measures they induce on the path space are mutually absolutely continuous. Since Gaussian processes are determined by their mean and covariance functions alone, it should be possible to formulate equivalence conditions involving these functions alone. This has been done in several cases. In particular, Shepp [6] has given the following necessary and sufficient conditions for a Gaussian process with covariance R and mean m to be equivalent to the Brownian motion on the interval $[0, T]$:

(i) $\min(s, t) - R(s, t)$ must be representable as $\int_0^s \int_0^t H(u, v)\,du\,dv$, where H is in L^2 and when considered as the kernel of a Hilbert–Schmidt operator on $L^2([0, T])$, its spectrum does not include the value one;

(ii) $m(t)$ must be representable as $\int_0^t f(u)\,du$, where f is in L^2.

Feldman [1] and Rozanov [4] consider stationary Gaussian processes X and Y with covariance functions $R(u)$ and $S(u)$, respectively, and show that if Y has a spectral density (this condition can be improved, see Feldman [2]), and if the process X has a spectral density f such that

$$0 < \liminf_{\lambda\to\infty} \lambda^2 f(\lambda) \leqq \limsup_{\lambda\to\infty} \lambda^2 f(\lambda) < \infty,$$

then X is equivalent to Y on $[0, T]$ if and only if $R(u) - S(u)$ has a derivative which is absolutely continuous on $(-T, T)$ with $\int_0^T \int_0^T (R - S)''(s - t)^2\,ds\,dt$ finite.

There is an obvious resemblance between these sets of conditions. Both require that the difference of the covariance functions should be signed distribution functions with densities in L^2, but whereas Shepp's result has the difference written as a specific definite integral, Feldman's result does not. On the other hand, Feldman's conditions are more restrictive than Shepp's in that the definite integral $\int_0^s \int_0^t H(u, v)\,du\,dv$ is not necessarily differentiable as a function of s. There is also a difference between the spectral condition on H in Shepp's theorem and the condition that the spectral function of Y be absolutely continuous in the Feldman result. In this paper we show why these similarities and subtle differences occur and equally important extend the results and the theory along the way.

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