*The Annals of Mathematical Statistics*
1969, Vol. 40, No. 6, 2118–2126

# WEAK QUALITATIVE PROBABILITY ON FINITE SETS

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**1. Introduction.** Recent works on intuitive and subjective probability [3, 9, 12, 13, 16, 23, 24] give axioms for a binary relation $\precsim$ ("is not more probable than") on an algebra that imply the existence of a probability measure $P$ on the algebra that strictly agrees [$A \precsim B \Leftrightarrow P(A) \leqq P(B)$] with $\precsim$. Kraft, Pratt, and Seidenberg [9] were the first to present necessary and sufficient conditions for strict agreement when the set $S$ of states is finite. Scott [16] rephrases these conditions.

This paper examines several finite-$S$ axiomatizations that result in partial rather than strict agreement. They take $\prec$ ("is less probable than") as primitive. In all cases $\prec$ is asymmetric so that at most one of $A \prec B$ and $B \prec A$ holds for any $A, B \subseteq S$.

In the next section we shall consider the case where $P$ almost agrees with $\prec$; $A \prec B \Rightarrow P(A) < P(B)$. Adams [1] gives necessary and sufficient conditions for this case. We shall also consider slightly stronger sufficient conditions that seem natural in the context of qualitative probability.

Section 3 presents even stronger conditions that yield a $P$ and a $\sigma \geqq 0$ such that $A \prec B \Leftrightarrow P(A) + \sigma(A) < P(B)$. In connection with this we shall present a theorem similar to Stelzer's [19] that gives necessary and sufficient conditions for a $P$ and a real number $0 \leqq \epsilon < 1$ such that $A \prec B \Leftrightarrow P(A) + \epsilon < P(B)$.

All our theorems are proved using a theorem of the alternative from linear algebra [2, 6, 21] whose broad applicability to relation–representation problems has been noted elsewhere [1, 5, 16, 22]. This theorem is in fact very efficient for uncovering conditions for numerical representation in linear systems, and it has been used in this way for the theorems of this paper. It is presented in Section 4 where proofs of two theorems of Section 3 are given.

Throughout, we define $A \sim B \Leftrightarrow$ (not $(A \prec B)$, not $(B \prec A)$). Our main divergence from the strict-agreement axioms [9, 16] is that we shall not assume that $\sim$ is transitive. This adds a dimension of reality to the theory of qualitative probability, and is an attempt to formalize the vagueness in judgment that Savage [14, 15] and others [7, 8, 18] have recognized. Now $A \sim B$ might have one of several interpretations, including the notion that $A$ and $B$ are equally probable, that there is not a definite feeling that $A$ is less probable than $B$ or vice versa, or that $A$ and $B$ are incomparable [7, 8]. Whatever the interpretation, an insistence that $\sim$ be transitive seems questionable. For example, suppose $A$, $B$, and $C$ are the events "it will rain here within the next 48 hours," "it will rain here within the next 49 hours," and "a Republican will be elected President in 1980." Then $A \prec B$, $A \sim C$, $B \sim C$ might well apply for an individual.

Received 1 October 1968; revised March 1969.