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# ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF FUNCTIONS OF ORDER STATISTICS[1]

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**1. Introduction.** Let $X_1, \cdots, X_N$ be i.i.d. uniform (0, 1) r.v.'s defined on a probability space $(\Omega, \mathfrak{A}, P)$. Let $F_N$ denote the empirical df, and let $X_{N1} \leqq \cdots \leqq X_{NN}$ denote the ordered $X_1, \cdots, X_N$. We wish to consider statistics of the form

$$T_N = N^{-1} \sum_{i=1}^N c_{Ni} g(X_{Ni}) \tag{1.1}$$

where $g$ is a specified function and $\{c_{Ni}: 1 \leqq i \leqq N, N \geqq 1\}$ is a set of specified constants.

REMARK. We may suppose the $X_i$'s to have an arbitrary continuous df $F$ provided we replace $g$ by $g(F^{-1})$.

We define inverses of df's to be left continuous; thus

$$F_N^{-1}(t) = \inf\{x: F_N(x) \geqq t\};$$

and we write $g \circ F_N^{-1}$ for the composed function $g(F_N^{-1})$. Note that $T_N = \int_0^1 g \circ F_N^{-1} \, d\nu_N$ when the signed measure $\nu_N$ puts mass $c_{Ni}/N$ at $i/N$ for $i = 1, \cdots, N$ and puts 0 mass elsewhere. Let $\nu$ denote a signed measure on (0, 1). (The signed measures $\nu_N$ will not be used, but $\nu$ is in some sense their limit.) For technical reasons to be displayed below, we bound ourselves away from 0 and 1 by an amount $\beta_N$ where $\beta_N \to 0$ as $N \to \infty$ at a rate to be specified later. Let

$$\mu_N = \int_{\beta_N}^{1-\beta_N} g \, d\nu \tag{1.2}$$

where $\int_\alpha^\beta \cdot \, d\nu = \int_{(\alpha,\beta]} \cdot \, d\nu$. Let $J_N$ be defined on (0, 1] by

$$J_N(t) = c_{Ni} \quad \text{for} \quad (i-1)/N < t \leqq i/N, \qquad 1 \leqq i \leqq N.$$

Let $I(t) = t$ be the identity function on [0, 1] and let $\int \cdot \, dI$ denote integrals with respect to Lebesgue measure. Let

$$T_N^* = N^{\frac{1}{2}}(T_N - \mu_N). \tag{1.3}$$

Define stochastic processes $\{L_N(t): 0 < t < 1\}$ for $N \geqq 1$ by

$$L_N(t) = N^{\frac{1}{2}}[g \circ F_N^{-1}(t) - g(t)]. \tag{1.4}$$

Then

$$T_N^* = S_N^* + \gamma_N \tag{1.5}$$

where

$$S_N^* = \int_{\beta_N}^{1-\beta_N} L_N \, d\nu \tag{1.6}$$

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