ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF
FUNCTIONS OF ORDER STATISTICS

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1. Introduction. Let $X_1, \ldots, X_N$ be i.i.d. uniform $(0, 1)$ r.v.'s defined on a
probability space $(\Omega, \mathcal{F}, P)$. Let $F_N$ denote the empirical df, and let $X_{\pi(1)} \leq \cdots \leq X_{\pi(N)}$ denote the ordered $X_1, \ldots, X_N$. We wish to consider statistics of
the form

$$T_N = N^{-1} \sum_{i=1}^{N} c_{N,i} g(X_{\pi(i)})$$

where $g$ is a specified function and $\{c_{N,i}: 1 \leq i \leq N, N \geq 1\}$ is a set of specified
constants.

Remark. We may suppose the $X_i$'s to have an arbitrary continuous df $F$
provided we replace $g$ by $g(F^{-1})$.

We define inverses of df's to be left continuous; thus

$$F^{-1}_N(t) = \inf \{x: F_N(x) \geq t\};$$

and we write $g \circ F_N^{-1}$ for the composed function $g(F_N^{-1})$. Note that
$T_N = \int_0^1 g \circ F_N^{-1} d\nu_N$ when the signed measure $\nu_N$ puts mass $c_{N,i}/N$ at $i/N$ for
$i = 1, \cdots, N$ and puts 0 mass elsewhere. Let $\nu$ denote a signed measure on $(0, 1)$.
(The signed measures $\nu_N$ will not be used, but $\nu$ is in some sense their limit.) For
technical reasons to be displayed below, we bound ourselves away from 0 and 1
by an amount $\beta_N$ where $\beta_N \to 0$ as $N \to \infty$ at a rate to be specified later. Let

$$\mu_N = \int_{\beta_N}^{1-\beta_N} g \, d\nu$$

where $\int_a^b \cdot d\nu = \int_{(a,b]} \cdot d\nu$. Let $J_N$ be defined on $(0, 1]$ by

$$J_N(t) = c_{N,i} \text{ for } (i - 1)/N < t \leq i/N, \quad 1 \leq i \leq N.$$

Let $I(t) = t$ be the identity function on $[0, 1]$ and let $\int \cdot dI$ denote integrals with
respect to Lebesgue measure. Let

$$T_N^* = N^{1/2}(T_N - \mu_N).$$

Define stochastic processes $L_N(t): 0 < t < 1$ for $N \geq 1$ by

$$L_N(t) = N^{1/2}[g \circ F_N^{-1}(t) - g(t)].$$

Then

$$T_N^* = S_N^* + \gamma_N$$

where

$$S_N^* = \int_{\beta_N}^{1-\beta_N} L_N \, d\nu.$$

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