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THE LAW OF THE ITERATED LOGARITHM FOR MIXING STOCHASTIC PROCESSES[1]

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1. Introduction. Let $\langle \xi_n, n = 1, 2, \cdots \rangle$ be a sequence of random variables centered at expectations with finite variances. Suppose that

$$s_N^2 = E(\sum_{n \leqq N} \xi_n)^2 \to \infty \qquad (N \to \infty) \tag{1}$$

$$s_{N+1}/s_N \to 1 \qquad (N \to \infty) \tag{2}$$

and that

$$s_{MN}^2 = E(\sum_{n=M+1}^{N} \xi_n)^2 = (s_N^2 - s_M^2)(1 + o(1)) \quad (\text{as } s_N^2 - s_M^2 \to \infty). \tag{3}$$

Let M_{ab} be the σ-algebra generated by the events $\{\xi_n < \alpha\}$, $a \leqq n \leqq b$. We say that the Borel-Cantelli Lemma holds for the process $\langle \xi_n \rangle$ if $\sum P(A_k) = \infty$ implies that $P(A_k \text{ i.o.}) = 1$ where $A_k \in M_{n_{k-1}n_k-1}(1 \leqq n_0 < n_1 < \cdots)$.

The standard proof of the law of the iterated logarithm yields the following

THEOREM 0. *Let $\langle \xi_n \rangle$ be any stochastic process satisfying* (1)–(3) *for which the Borel-Cantelli Lemma holds. Suppose that uniformly in M and x*

$$P(s_{MN}^{-1} \sum_{n=M+1}^{N} \xi_n < x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt + O((\log s_{MN})^{-1-\eta}), \eta > 0, \tag{4}$$

and that for some constants $C > 0$, $0 < \rho_N = O((\log \log s_N)^{\frac{1}{2}})$ and ϵ sufficiently large

$$P(\max_{1 \leqq n \leqq N} \sum_{k \leqq n} \xi_k > \epsilon) \leqq CP(\sum_{k \leqq N} \xi_k > \epsilon - \rho_N s_N). \tag{5}$$

Moreover, suppose that (5) *holds with ξ_n replaced by $-\xi_n$. Then*

$$P(\limsup_{N \to \infty} (2s_N^2 \log \log s_N^2)^{-\frac{1}{2}} \sum_{n \leqq N} \xi_n = 1) = 1. \tag{6}$$

In short the law of the iterated logarithm holds for any process for which the Borel-Cantelli Lemma, the central limit theorem with a reasonably good remainder and a certain maximal inequality are valid. The proof of Theorem 0 can be found in Loève [4, pages 260–263] (see also [1], [5]) where instead of the exponential bounds we use the fact that for $\tau > 0$

$$P(\sum_{n=M+1}^{N} \xi_n > \tau s_{MN}) = (2\pi)^{-\frac{1}{2}}\tau^{-1} \exp(-\tfrac{1}{2}\tau^2)(1 + \theta\tau^{-2}) + O((\log s_{MN})^{-1-\eta})$$

with $0 < \theta < 1$. This follows from (4) and the well-known [1, page 175] estimate

$$\int_x^{\infty} e^{-\frac{1}{2}t^2}\, dt = x^{-1} \exp(-\tfrac{1}{2}x^2)(1 + \theta x^{-2}).$$

Moreover, we choose n_k to be the largest integer n with $s_n \leqq c^k$, where $c > 1$ is the constant occurring in [4, page 261], (s_n is not assumed to be monotone).

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