The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 318–320

A MONOTONICITY PROPERTY OF THE DISTRIBUTION OF THE STUDENTIZED SMALLEST CHI-SQUARE[1]

BY KHURSHEED ALAM[2]

Indiana University

1. Main theorem. Let $X_1, \cdots, X_k$ be k i.i.d. random variables, each having a gamma distribution with m degrees of freedom. The random variable

$$X = \min(X_1/X_k, \cdots, X_{k-1}/X_k), \tag{1.1}$$

is called the Studentized smallest chi-square. Its cumulative distribution function (cdf) is given by

$$G_m(x) = 1 - \int_0^\infty (1 - F_m(xy))^{k-1}\, dF_m(y) \tag{1.2}$$

where $F_m(y) = \{\Gamma(m)\}^{-1} \int_0^y x^{m-1} e^{-x}\, dx$ denotes the incomplete gamma function. Clearly, $G_m(1) = (k-1)/k$. A monotonicity property of the cdf of X, which has some applications, is given by the following theorem.

THEOREM 1.1. *For* $m > 1$, $G_m(x)$ *is increasing (decreasing) in* m *for* $x > (<) 1$.

PROOF. Let Y denote a random variable with cdf $F_m(y)$. For $m > 1$ let

$$X = F_m(cY) \tag{1.3}$$

where $c > 0$ is a constant. The probability density function of X is given by

$$\begin{aligned} g_m(x) &= (f_m(F_m{}^{-1}(x)/c))/(cf_m(F_m{}^{-1}(x))) \\ &= c^{-m} \exp((c-1)F_m{}^{-1}(x)/c), \qquad 0 < x < 1, \end{aligned} \tag{1.4}$$

where $f_m(x) = x^{m-1}e^{-x}/\Gamma(m)$ and $F_m{}^{-1}(x)$ denotes the inverse function of $F_m(x)$. For $r > 0$ let

$$\begin{aligned} A(x) &= f_m(F_m{}^{-1}(x)) - f_{m+r}(F_{m+r}^{-1}(x)) \\ &= F_{m-1}(F_m{}^{-1}(x)) - F_{m+r-1}(F_{m+r}^{-1}(x)) \end{aligned} \tag{1.5}$$

and

$$B(x) = \log g_{m+r}(x) - \log g_m(x). \tag{1.6}$$

Then

$$\begin{aligned} dB(x)/dx &= (c-1)c^{-1}(1/f_{m+r}(F_{m+r}^{-1}(x)) - 1/f_m(F_m{}^{-1}(x))) \\ &= (c-1)c^{-1}A(x)/(f_m(F_m{}^{-1}(x))f_{m+r}(F_{m+r}^{-1}(x))). \end{aligned} \tag{1.7}$$

It is shown below that $A(x)$ is nonnegative. Therefore, from (1.7) we have that $B(x)$ is nondecreasing (nonincreasing) in x for $c > (<) 1$. This result will be used in the sequel.

Received November 12, 1968; revised July 29, 1969.
[1] This work was supported in part by NSF Grant GP7496 at Indiana University.
[2] Now at Clemson University.