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ON THE CONSTRUCTION OF ALMOST UNIFORMLY CONVERGENT RANDOM VARIABLES WITH GIVEN WEAKLY CONVERGENT IMAGE LAWS[1]

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1. Introduction. Let S be an arbitrary metric space, with distance function d, and let $\mathscr{S}$ be its Borel σ-algebra. Denote by $\mathscr{P}(S)$ the class of all probability distributions on $(S, \mathscr{S})$. A net $(P_\gamma)_{\gamma\in\Gamma}$ of probabilities $P_\gamma \in \mathscr{P}(S)$ is said to converge weakly to a probability $P \in \mathscr{P}(S)$ if $P(f) = \lim_\gamma P_\gamma(f)$ for each real-valued bounded continuous function f on S; here $P(f) = \int f\,dP$, $P_\gamma(f) = \int f\,dP_\gamma$. Let $\mathscr{P}_s(S)$ denote the subclass of $\mathscr{P}(S)$ consisting of those probabilities P for which there exists a separable subset of S in $\mathscr{S}$ of P-probability one. $\mathscr{P}_s(S)$ includes the so-called tight probabilities i.e. probabilities P such that $\sup\{P(K): K \text{ compact}\} = 1$ ([5] page 29). The chief result of this paper is stated in the following.

THEOREM 1. *Let (S, d) be a metric space and let $(P_\gamma)_{\gamma\in\Gamma}$ be a net of probabilities $P_\gamma \in \mathscr{P}(S)$ converging weakly to a probability $P \in \mathscr{P}_s(S)$. Then there exists a probability space $(\Omega, \mathscr{B}, \mu)$ and $\mathscr{B}$–$\mathscr{S}$ measurable, S-valued functions X and $X_\gamma(\gamma\in\Gamma)$ defined on Ω such that the distributions μX^{-1} of X and μX_γ^{-1} of X_γ are respectively P and $P_\gamma(\gamma \in \Gamma)$ and such that X_γ converges to X almost uniformly.*

One sometimes ([1], [8]) has occasion to consider the weak convergence of probability distributions P_γ which are defined only on certain sub-σ-algebras of $\mathscr{S}$, and it is therefore of interest to know that the requirement in Theorem 1 that the P_γ belong to $\mathscr{P}(S)$ can be weakened. To make this precise, let us say that a net $(P_\gamma)_{\gamma\in\Gamma}$ of probabilities P_γ defined on sub-σ-algebras $\mathscr{A}_\gamma$ of $\mathscr{S}$ converges weakly to a probability $P \in \mathscr{P}(S)$ if $\lim_\gamma \bar{P}_\gamma(f) = P(f) = \lim_\gamma \underline{P}_\gamma(f)$ for each real-valued bounded continuous function f on S; here $\bar{P}_\gamma$ and $\underline{P}_\gamma$ denote respectively the upper and lower probabilities associated with P_γ:

$$\bar{P}_\gamma(f) = \inf\{P_\gamma(g): f \leqq g,\ P_\gamma(g) \text{ defined}\}$$

$$\underline{P}_\gamma(f) = \sup\{P_\gamma(g): f \geqq g,\ P_\gamma(g) \text{ defined}\}$$

(for equivalent formulations of this definition see Theorem 1 of [8]). It is clear that this definition of weak convergence reduces to the usual one if all the $\mathscr{A}_\gamma$ equal $\mathscr{S}$. Let $\mathscr{S}_0$ denote the sub-σ-algebra of $\mathscr{S}$ generated by the open balls of S. We then have the following extension of Theorem 1:

THEOREM 2. *Let $S, \mathscr{S}$, and $\mathscr{S}_0$ be defined as above and let $(P_\gamma)_{\gamma\in\Gamma}$ be a net of probabilities P_γ, defined on σ-algebras $\mathscr{A}_\gamma$ containing $\mathscr{S}_0$ and contained in $\mathscr{S}$, which*

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