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A LIMIT THEOREM FOR CONDITIONED RECURRENT RANDOM WALK ATTRACTED TO A STABLE LAW[1]

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1. Introduction. Consider an ensemble of independent particles whose motion describes a random walk on Z^d, the d-dimensional lattice of integers. If **A** is an arbitrary subset of Z^d and the random walk is assumed recurrent (consequently $d \leqq 2$), then as time passes it becomes increasingly unlikely that any given particle has avoided **A**. Suppose, however, that at each stage attention is restricted to only those particles whose past history is such that **A** has been avoided. Then it is of interest to investigate the possible distortive effects of this conditioning on the asymptotic behavior of the particle motion. Suppose **A** is finite and $\tilde{g}_{\mathbf{A}}(0) \neq 0$ (the function $\tilde{g}_{\mathbf{A}}(x)$ of potential-theoretic interest is defined below and the connection between this condition and the motion of the random walk established) and suppose that the underlying distribution F governing the particle transitions is attracted to a stable law $G_\alpha(1 \leqq \alpha \leqq 2$ is the index of the stable law). The principal result of the paper (Theorem 2.1) states that the conditional distribution of the particles whose past motion has avoided the set **A** is also attracted to a limit distribution H_α. Except for the case $d = 1$ with G_α a Cauchy distribution and the case $d = 2$ with G_α a normal distribution, the distributions G_α and H_α are in general different. For $d = 1$ and $\alpha = 2$, under certain further restrictions on **A**, G_α turns out to be a two-sided Rayleigh distribution. It is the case, however, that the same constants normalizing the particle position may be used in the statement of the attraction of the conditioned motion to H_α as in the statement of the attraction of the unconditioned motion to the stable law G_α. In preparation we first review some basic definitions and record some preliminary facts about recurrent lattice random walk.

We let $p: Z^d \times Z^d \to [0, 1]$ be the transition function of the random walk. Thus,

(i) $$p(x_1, x_2) = p(0, x_2 - x_1) \quad \text{for} \quad x_1, x_2 \in Z^d$$

(ii) $$\textstyle\sum_{x \in Z^d} p(0, x) = 1,$$

and we inductively define

$$p_n(x_1, x_2) = \textstyle\sum_{y \in Z^d} p_{n-1}(x_1, y) p(y, x_2) \quad \text{for} \quad n = 2, 3 \cdots.$$

An underlying probability space (Ω, P, B) is assumed to have been constructed, on which a sequence of independent random variables X_i, $i = 1, 2, \cdots$ (the increments of the random walk) are defined, such that $P[X_i = x] = p(0, x)$, $i = 1, 2, \cdots$.

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