The Annals of Mathematical Statistics
1970, Vol. 41, No. 2, 723–728

THE DISTRIBUTION OF THE RATIOS OF MEANS TO THE SQUARE ROOT OF THE SUM OF VARIANCES OF A BIVARIATE NORMAL SAMPLE

BY S. A. PATIL AND S. H. LIAO

Tennessee Technological University

1. Introduction. Let (X_i, Y_i) $i = 1, 2, \cdots, m$, be independent observations on a random vector (X, Y) which has a bivariate normal distribution with

$$EX = EY = 0, \qquad EX^2 = EY^2 = \sigma^2, \qquad EXY = \rho\sigma^2.$$

Let

$$\bar{X} = m^{-1}\sum_{i=1}^m X_i, \qquad \bar{Y} = m^{-1}\sum_{i=1}^m Y_i, \qquad s_1{}^2 = m^{-1}\sum_{i=1}^m (X_i - \bar{X})^2,$$

$$s_2{}^2 = m^{-1}\sum_{i=1}^m (Y_i - \bar{Y})^2.$$

Recently Siddiqui [6] has considered the distribution of $(m-1)^{\frac{1}{2}}\bar{X}/s_1$, $(m-1)^{\frac{1}{2}}\bar{Y}/s_2$. For $m > 3$, he obtained asymptotic results. We define $Z = ms_1{}^2 + ms_2{}^2$, $s^2 = (2m-1)^{-1}Z$, so that s^2 is an unbiased estimator of σ^2 based on both X and Y observations. In this note we consider the distribution of (T_1, T_2), where $T_1 = m^{\frac{1}{2}}s^{-1}\bar{X}$, $T_2 = m^{\frac{1}{2}}s^{-1}\bar{Y}$. It is noted that (T_1, T_2) are independent of the scale parameter. We have obtained the probability density function (pdf) of (T_1, T_2) and the distribution function of (T_1, T_2). Also marginal and limiting distributions are discussed.

2. The probability density function of Z. The following lemma is used to determine the pdf of Z.

LEMMA. *Let* (X_i, Y_i), $i = 1, 2, \cdots, m$; $m > 3$ *be the observations from the bivariate normal distribution with the zero means, correlation coefficient* ρ *and common variance* σ^2; *then the distribution of* Z, *defined in Section* 1, *can be expressed as the distribution function of* $[U_1(1+\rho)\sigma^2 + U_2(1-\rho)\sigma^2]$, *where* U_1, U_2 *are independent and identically distributed chi-square random variables with* $(m-1)$ *degrees of freedom.*

Using Lemma 2 of [1] it can be easily shown that

$$(1) \qquad M_Z(t) = E\,e^{tZ} = [1-2t(1+\rho)\sigma^2]^{-\frac{1}{2}(m-1)}[1-2t(1-\rho)\sigma^2]^{-\frac{1}{2}(m-1)}$$

which is the same as moment generating function of $[U_1(1+\rho)\sigma^2 + U_2(1-\rho)\sigma^2]$. The distribution of Z does not depend on the means of (X_i, Y_i).

Received June 4, 1969; revised October 8, 1969.