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CONVEX CONES AND FINITE OPTIMALS

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1. Introduction. Among the continuous payoff kernels $K(x, y)$ on the unit square for an infinite zero-sum two person game, the separable kernels, the generalized convex kernels and certain analytic kernels are known to possess optimal mixed strategies for the two players with finite spectrum [5]. It is known that even among C^∞ kernels we can have very pathological optimal mixed strategies as their unique optimals [4]. Thus the problem of classifying kernels with finite optimals is unresolved. Here an attempt is made to look at this problem from the topological viewpoint. The binding geometric object between kernels and strategies could be chosen as the cone generated by functions $h_\alpha(x) = K(x, \alpha)$ where we fix α and view $K(x, \alpha)$ as a function of x. Some of the properties of the cones are reflected in the finiteness of the spectrum for an optimal strategy for a player. Similar versions could be stated for the other player, by considering a related kernel where now the second player becomes the maximizer. Further, these cones in certain other topologies also characterize extreme optimals for a class of games.

Preliminaries. Let X, Y be compact metric spaces and $K(x, y) > 0$ be a continuous payoff on $X \times Y$. Let E_X, E_Y be the Banach space of continuous functions on X and Y with their supremum norm ($\|\cdot\|$). Let C be the closure of the convex cone in E_X generated by functions $h_\alpha(x)$, where $h_\alpha(x) = K(x, \alpha)$, $\alpha \in Y$. Let K be the cone of nonnegative functions in E_X. Let E_0 be the linear manifold C–C and $\bar{E}_0$ its closure. By a positive operator we mean a linear operator A from $\bar{E}_0 \to E_X$ which maps the cone C into K. We would call the image of the cone C under A the range cone. The following is the main theorem.

Theorem 1. *Let A be a positive operator from $\bar{E}_0 \to E_X$ continuous on the cone C. If A is isometric on C and if the range cone has a relative interior point in the closed linear manifold spanned by this cone, then player II has always an optimal mixed strategy whose spectrum is finite. If the cone C itself possesses an interior point relative to $\bar{E}_0$ then the conditions are trivially satisfied for the identity map and in this case the kernel is separable and both players have optimal mixed strategies with finite spectrum.*

Proof. Let P_Y denote the set of all probability measures on the Borel sets of Y. P_Y as a subset of $E_Y{}^*$ (the dual of E_Y) is compact metric in its weak topology [6]. Further by Helly's theorem

$$\tau : v \to \int_Y K(x, y)\, dv(y)$$

is continuous from P_Y into E_X. Thus $\tau(P_Y) = B$ is compact in E_X. Trivially it is convex. Let T be the convex cone generated by B, i.e. $T = \{\lambda f : \lambda \geqq 0, f \in B\}$. We

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