The Annals of Mathematical Statistics
1970, Vol. 41, No. 2, 646–647

AN ADDENDUM TO "STOCHASTIC APPROXIMATION AND NONLINEAR REGRESSION"

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The purpose of this note is to point out a strengthening of results on asymptotic normality given in Albert and Gardner [1]. In brief, probability one convergence of the method of differential corrections can be used to replace global limit requirements by local ones. The authors wish to thank Professor Vaclav Fabian for bringing this point to their attention. The approach was evidently first taken by Hodges and Lehmann [2] in their study of the large sample distribution of the Robbins–Monro procedure.

In what follows the undefined quantities are those in [1]. We will be discussing only Theorem 5.2, but it will be clear that corresponding alterations should also be made in certain theorems which precede it.

Consider, then, the estimate $t_n = t_n^{(2)}$ of Theorem 5.2 with a gain constant

$$A_2 > \tfrac{1}{2}.$$

The assumptions imply those of Theorem 2.1 or, more precisely, its corollary which is Theorem 2.3, Condition 3. Consequently, $t_n \to \theta$ a.s. as $n \to \infty$. Let $\delta > 0$ be given. By Egorov's theorem there is an N, depending on δ but not on points in the sample space, such that $|t_n - \theta| < \delta$ for all $n \geqq N$ with probability at least $1-\delta$. For $n \geqq N$ define new regression functions F_n' with derivatives

$$\begin{aligned} \dot{F}_n'(x) &= \dot{F}_n(\theta-\delta) && \text{for} \quad x < \theta - \delta \\ &= \dot{F}_n(x) && \text{for} \quad x \in J_\delta = [\theta-\delta, \theta+\delta] \\ &= \dot{F}_n(\theta+\delta) && \text{for} \quad x > \theta+\delta \end{aligned}$$

where x is further restricted to belong to the original interval J. For $n < N$, set $F_n' = F_n$ for all $x \in J$. Define new estimates $\{t_n'\}$ in terms of $\{F_n'\}$. Then

$$P\{t_n = t_n' \text{ for all } n \geqq N\} > 1 - \delta.$$

Now the primed problem obeys all the conditions of Theorem 5.2 with

$$b_n' = \inf_{x \in J_\delta} |\dot{F}_n'(x)| = b_n \inf_{x \in J_\delta} g_n(x).$$

We have uniformly in $x \in J_\delta$

$$g_n'(x) = \frac{b_n}{b_n'} g_n(x) \to \frac{g(x)}{\inf_{x \in J_\delta} g(x)} = g'(x) \qquad (n \to \infty)$$

with the appropriate replacement of x by the boundary points when x is outside the interval. By the conclusion of Theorem 5.2, $S_n(t_n' - \theta)$ is asymptotically normal,

Received May 8, 1969.