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A SYSTEM OF MARKOV CHAINS WITH RANDOM LIFE TIMES[1]

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0. Introduction. The purpose of this paper is to investigate the limiting properties of random variables associated with a system of random processes. The system is described as follows. At each discrete integer time $n \geqq 0, M_n$ particles enter a denumerable set of states Λ at a given state denoted by $(0, 0)$. Assume $\{M_n, n \in I\}$ to be a sequence of independent Poisson variables with common mean λ. (Here and throughout, I denotes the set of nonnegative integers.) Moreover, at each integer time $n \geqq 1$, each particle already in the system may undergo a transition independently of the other particles and independently of $\{M_n, n \in I\}$. A particle which entered the system at time $k \leqq n$, moves according to the probability law of $Z(n-k)$ where $\{Z(n), n \in I\}$ is a random process described below.

1. Preliminaries. Let $\{X(n), n \in I\}$ be an irreducible aperiodic Markov chain having state space Γ, taken to be the nonnegative integers, and having stationary transition probabilities $P(x, y)$. Let $P_n(x, y)$ denote the n-step transition probabilities and $P_n(x, B) = \sum_{y \in B} P_n(x, y)$ for sets $B \subseteq \Gamma$. Let $\{Y(n), n \in I\}$ be a random process with state space $\{0, 1\}$, independent of $\{X(n), n \in I\}$. Let $p(n) = P[Y(n) = 0]$, $p = \{p(n), n \in I\}$, and assume $Y(n) = 1$ implies $Y(n+1) = 1$ for each $n \in I$. Thus $p(n) \geqq p(n+1)$ and $\pi \equiv \lim_{n \to \infty} p(n)$ exists. Define $Z(n) = (X(n), Y(n))$. The process $\{Z(n), n \in I\}$ has state space $\Lambda = \{(x, y) : x \in \Gamma, y = 0 \text{ or } 1\}$. The independence assumption of the introduction means that the sequence $\{M_n, n \in I\}$ is independent of the processes $\{X(n), n \in I\}$ and $\{Y(n), n \in I\}$. One can think of the transition of a particle in its y coordinate from state 0 to state 1 as the death of this particle. Accordingly, transitions of the process $\{Z(n), n \in I\}$ through states of the form $(x, 0), x \in \Gamma$, can be thought of as the transitions of a particle according to the law of the Markov chain while the particle is still alive. Two special cases of the $Y(n)$ process are of interest. If $\pi = 1$ no deaths occur and $Z(n)$ is Markov with transition probabilities $P(x, y)$. If for some $n_0 \in I, n_0 > 0, p(n) = 1$ if $n \leqq n_0$ and $p(n) = 0$ for $n > n_0$ the particles have fixed life times. In this case it will be seen that the system of live particles attains a macroscopic equilibrium. See Section 2 for details.

In what follows, $B \subset \Gamma$ is assumed finite and, to avoid trivialities, not to include state 0. Let V_B^r denote the time of rth visit to B by $X(n)$ and $N_k(B)$ the occupation time of B by $X(n)$ to time k. Formally,

$$V_B^1 = \min\{n : X(n) \in B\}, \qquad V_B^r = \min\{n : n > V_B^{r-1}, X(n) \in B\}$$

where if for some integer $r > 0$, $X(n) \notin B$ for all $n > V_B^{r-1}$, take $V_B^r = \infty$. Further $N_k(B) = \sum_{j=1}^k \delta_B(X_j)$ where $\delta_B(x) = 1$ (or 0) if $x \in B$ (or $x \notin B$). Let $N(B) = \lim_{k \to \infty} N_k(B)$ whether finite or infinite. Probabilities for the random variables V_B^r,

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