*The Annals of Mathematical Statistics*
1970, Vol. 41, No. 2, 567–575

# FURTHER RESULTS ON MINIMUM VARIANCE UNBIASED ESTIMATION AND ADDITIVE NUMBER THEORY

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**1. Introduction.** Patil [4] and [5] has investigated problems of existence of minimum variance unbiased (MVU) estimators of parametric functions for the univariate and multivariate power series distributions (PSD) in terms of the number theoretic structure of their respective ranges. A uniform technique of obtaining the MVU estimators, when they exist, has also been provided. We propose to present here certain additional investigations in this area. Introduction of the $D$ and $E$ sets helps develop an interestingly neat picture of the seemingly complicated structural situations and problems. We shall use essentially the same notation and terminology as in Patil [4] and [5]. The next section provides supplemental notation and terminology and quotes certain results that will be used in the text.

**2. Notation and terminology.** (i) Let $I_s$ denote the $s$-fold cartesian product of the set $I$ of nonnegative integers with itself. In general, let $\prod_{i=1}^{s} T_i$ denote the cartesian product of the $s$ sets $T_1, T_2, \cdots, T_s$.

(ii) Let $A \subset I_s$ and $B \subset I_s$. $A$ is called a *basis* for $B$ if the $n$-fold sum of $A$ defined by $n[A] = \{\sum_{i=1}^{n} \mathbf{a}_i, \mathbf{a}_i \in A\}$ is equal to $B$ for some $n$. In such a case, $n$ is called an *order* of the *basis* $A$ for $B$.

(iii) Let $T = \prod_{i=1}^{s} T_i$, $T_i \subset I$. The displaced set $D(T)$ is defined as $D(T) = T - \{(\min(T_1), \min(T_2), \cdots, \min(T_s))\}$. Thus, $D(T)$ is the difference between $T$ and the singleton $\{(\min(T_1), \min(T_2), \cdots, \min(T_s))\}$. It is clear that $D(T) = D(\prod_{i=1}^{s} T_i) = \prod_{i=1}^{s} D(T_i)$.

(iv) For an arbitrary subset $T$ of $I_s$ and $\mathbf{x} \in I_s$, $D_x(T)$ denotes the difference set $T - \{\mathbf{x}\}$.

(v) Let $T \subset I_s$. $\mathbf{a} \in T$ is called a lower boundary point (lbp) of $T$ if there is no $\mathbf{x} \in T$, $\mathbf{x} \neq \mathbf{a}$, such that the $i$th components satisfy the inequality $x_i \leqq a_i$, for $i = 1, 2, \cdots, s$. The lower boundary (LB) of $T$ is defined to be the set consisting of the lbp's of $T$ and is denoted by LB($T$). It is clear that, if $\mathrm{LB}(T) \ni \mathbf{a} \neq \mathbf{b} \in \mathrm{LB}(T)$, then $a_i < b_i$ and $a_j > b_j$ for *some* pair $i \neq j$.

(vi) Let $\mathbf{0} \in A \subset I_s$. Then, the Kvarda–Schnirelmann density $d(A)$ of the set $A$ is defined to be

$$d(A) = \underset{R}{\mathrm{glb}} \frac{A(R)}{I_s(R)}$$

where the glb is taken over all finite subsets $R$ of $I_s$, excluding $\{0\}$ and the empty set, with the property that if $\mathbf{r} \in R$, $\mathbf{x} \in I_s$ and $x_i \leqq r_i$, $i = 1, 2, \cdots, s$, then $\mathbf{x} \in R$ and

Received December 2, 1968.

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