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FUNCTIONS OF PROCESSES WITH MARKOVIAN STATES—III

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1. Introduction. Let $\{Y_k\}$ be a stochastic process where either $k = 1, 2, \cdots$ or $k = 0, \pm 1, \pm 2, \cdots$. Suppose there exist a time n and a state ε such that $P(Y_n = \varepsilon) > 0$. In this case, the rank at time n of the state ε is defined in [3], although the notion was first considered by Gilbert in [5]. The definition is such that a state of rank 1 is Markovian.

Let $\{X_k\}$ be a second stochastic process indexed as $\{Y_k\}$. Gilbert [5] proved (but stated in far less generality) that if $\nu_n(\varepsilon)$ and $\mu_n(\delta)$ are the ranks at time n of the states ε of $\{Y_k\}$ and δ of $\{X_k\}$ respectively, and if $Y_n = f(X_n)$, then

$$\nu_n(\varepsilon) \leqq \textstyle\sum_{f_n(\delta)=\varepsilon} \mu_n(\delta), \tag{1.1}$$

Dharmadhikari [1] considered the case of stationary $\{Y_k\}$ with all states of finite rank and found an additional condition in order to guarantee $Y_k = f(X_k)$ for $\{X_k\}$ stationary and Markovian. The present authors [3] provided an example showing Dharmadhikari's result cannot be obtained without some condition beyond finiteness of rank.

In the present paper we extend the definition of rank by eliminating the condition $P(Y_n = \varepsilon) > 0$ and generalize Gilbert's, Dharmadhikari's, and our own results.

Section 2 contains the extension of the definition of rank. Sections 3 and 4 contain, respectively, the extensions of Gilbert's and Dharmadhikari's results. The extension of the Dharmadhikari-type result to the nonstationary case is discussed in Section 5. Section 6 contains an outline of the extension of the work of the present authors [4].

2. The extended definition of rank. Let U_k be the state space of $\{Y_k\}$ at time k and set $V_n = \cdots \times U_{n-2} \times U_{n-1}$ and $W_n = U_{n+1} \times U_{n+2} \times \cdots$. Let $\mathscr{A}_n$, $\mathscr{S}_n$ and $\mathscr{T}_n$ be, respectively, the σ-algebras of measurable subsets of U_n, V_n and W_n. Assume all $\mathscr{A}_i$ (and hence $\mathscr{S}_n$ and $\mathscr{T}_n$) are separable. For $A \in \mathscr{A}_n$, $S \in \mathscr{S}_n$ and $T \in \mathscr{T}_n$ let

$$P_n(S, A, T) = P((\cdots, Y_{n-2}, Y_{n-1}) \in S, \quad Y_n \in A, \quad (Y_{n+1}, Y_{n+2}, \cdots) \in T).$$

Let $Q_n(A) = P_n(V_n, A, W_n)$ and

$$p_n(S, \varepsilon, T) = \frac{dP_n(S, \cdot, T)}{dQ_n}(\varepsilon).$$

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