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A DUALITY BETWEEN AUTOREGRESSIVE AND MOVING AVERAGE PROCESSES CONCERNING THEIR LEAST SQUARES PARAMETER ESTIMATES[1]

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1. Introduction. The methods employed in least squares parameter estimation for moving average (MA) processes differ from those appropriate for autoregressive (AR) processes, as only the latter are linear in the parameters. There is nevertheless an interesting duality between these two classes of time series models: if AR and MA series, each of the same order and with the same parameter values, are generated from the same sequence of errors, then to a close approximation the least squares estimates calculated from the MA series will *under*estimate the true parameter values by the same amount that those determined from the AR series will *over*-estimate them. This relation is established in Section 3 via a linear approximation of the moving average errors (considered as functions of the parameters for a given series) in a neighborhood of the true parameter values. In Section 4 the large-sample distribution of the estimates in any MA process is then obtained as a direct consequence of this relation and of known results for AR processes, and some properties of the least squares estimates in both classes of processes are examined.

Let us introduce notation and terminology to be employed. A sequence $\{x_t\}$ follows an AR *process of order p* if it is governed by the relation

$$x_t = \sum_{j=1}^{p} \theta_j x_{t-j} + a_t \tag{1.1}$$

where the $\{a_t\}$ are $N(0, \sigma^2)$ and independent, and the parameters $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$ are such that the roots of the auxiliary equation

$$\theta(u) = 1 - \theta_1 u - \cdots - \theta_p u^p = 0 \tag{1.2}$$

lie outside the unit circle (the set of all $\boldsymbol{\theta}$ satisfying this condition will be referred to as the *admissible parameter space*). Following [3] we may define a *backward shift operator B* by the relation $Bw_t = w_{t-1}$ for any sequence $\{w_t\}$, and equation (1.1) may then be written

$$\theta(B)x_t = a_t \tag{1.3}$$

where $\theta(B) = (1 - \sum_{j=1}^{p} \theta_j B^j)$ analogous to (1.2).

In the MA process the recursive relationship is in terms of the deviates $\{a_t\}$ rather than the observations themselves; a sequence $\{y_t\}$ is a moving average of order p if

$$y_t = -\sum_{j=1}^{p} \theta_j a_{t-j} + a_t = \theta(B)a_t \tag{1.4}$$

with $\{a_t\}$ and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$ as before.

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