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A MARTINGALE DECOMPOSITION THEOREM[1]

BY GORDON SIMONS

University of North Carolina

Let Z be a random variable with $E|Z| < \infty$ and define recursively

$$Z_0 = EZ, \qquad Z_n = E^{\mathscr{F}_n}Z, \tag{1}$$

where

$$\mathscr{F}_n = \mathscr{B}(Z_{n-1}, I(Z \geqq Z_{n-1})) \quad \text{for} \quad n = 1, 2, \cdots.^2 \tag{2}$$

The Z_n sequence constitutes a martingale decomposition of Z in the sense of the following

THEOREM.

(i) $Z_0, Z_1, \cdots, Z_n, \cdots, Z$ *is a martingale.*

(ii) *The conditional distribution of* Z_n *given* Z_{n-1} *is a one or two point distribution* a.s. *for* $n = 1, 2, \cdots$.

(iii) $Z_n \to Z$ a.s. *as* $n \to \infty$.

PROOF. It is useful to define a closely related sequence by

$$Y_0 = EZ, \qquad Y_n = E^{\mathscr{G}_n}Z, \tag{3}$$

where

$$\mathscr{G}_n = \mathscr{B}(Y_i, I(Z \geqq Y_i); i = 0, \cdots, n-1) \quad \text{for} \quad n = 1, 2, \cdots. \tag{4}$$

We shall show that

$$\bar{\mathscr{F}}_n = \bar{\mathscr{G}}_n \tag{5}$$

from which we may conclude (i) (cf., [1] page 293) and

$$Y_n = Z_n \text{ a.s.} \quad \text{for} \quad n = 0, 1, \cdots. \tag{6}$$

To show (5), it suffices to show for $0 \leqq j < k$ that

$$Z \geqq Y_j \quad \text{if, and only if,} \quad Y_k \geqq Y_j \quad \text{a.s.} \tag{7}$$

and

$$Y_j \text{ is measurable with respect to } \bar{\mathscr{B}}(Y_k). \tag{8}$$

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[2] We shall assume that everything is defined on a basic probability space (Ω, $\mathscr{F}$, P). For an arbitrary event $A \in \mathscr{F}$ and arbitrary random vector W, we denote $I(A)$ and $\mathscr{B}(W)$ as the indicator function (taking the value 1 on A and 0 off A) and the σ-field generated by W respectively. $\bar{\mathscr{B}}(W)$ will refer to the smallest σ-field containing $\mathscr{B}(W)$ and the null sets of $\mathscr{F}$.