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DISTRIBUTIONS CONNECTED WITH A MULTIVARIATE BETA STATISTIC

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1. Introduction. Let $A_r(p \times p)(r = 1, \cdots, q)$ and $B(p \times p)$ be independently distributed according to Wishart (Σ_1, m_r) and Wishart (Σ_2, n) respectively. Let

$$L_r = E^{-\frac{1}{2}} A_r (E^{-\frac{1}{2}})', \tag{1.1}$$

$E^{\frac{1}{2}}$ being a lower triangular matrix such that $E^{\frac{1}{2}}(E^{\frac{1}{2}})' = E = \sum_{r=1}^{q} A_r + B$. The purpose of this paper is to derive the joint density of $L_1, L_2, \cdots, L_q$, an asymptotic distribution for $\prod_{r=1}^{q} |L_r|$ and an asymptotic distribution for $|I - \sum_{r=1}^{q} L_r|$.

2. Preliminary results.

LEMMA 1. *If R and S are two positive definite symmetric matrices of order p, then* (*Constantine* (1963)),

$$\int_{E>0} \exp(-\tfrac{1}{2} \operatorname{tr} RE) |E|^{\alpha - \frac{1}{2}(p+1)} C_K(SE)\, dE = \Gamma_p(\alpha) |R|^{-\alpha} (\alpha)_K C_K(R^{-1}S). \tag{2.1}$$

LEMMA 2. *If R is a positive definite symmetric matrix of order p, then* (*Constantine* (1963))

$$\int_{0<Z<I} |Z|^{\alpha - \frac{1}{2}(p+1)} |I - Z|^{b - \frac{1}{2}(p+1)} C_K(RZ)\, dZ = (\Gamma_p(\alpha)\Gamma_p(b)/\Gamma_p(\alpha+b))((\alpha)_K/(\alpha+b)_K\, C_K(R). \tag{2.2}$$

LEMMA 3. *If R is a positive definite symmetric matrix of order p, then*

$$\int \cdots \int_{0<L_r<I,\, 0<\sum_{r=1}^{q} L_r<I} \prod_{r=1}^{q} |L_r|^{\alpha_r - \frac{1}{2}(p+1)} |I - \sum_{r=1}^{q} L_r|^{b - \frac{1}{2}(p+1)} \cdot C_K(R \sum_{r=1}^{q} L_r) \prod_{r=1}^{q} dL_r = (\Gamma_p(\alpha) \prod_{r=1}^{q} \Gamma_p(\alpha_r)/\Gamma_p(\alpha+b))((\alpha)_K/(\alpha+b)_K) C_K(R) \tag{2.3}$$

where $\alpha = \sum_{r=1}^{q} \alpha_r$.

PROOF. Let $\phi(L_1, \cdots, L_q) = \prod_{r=1}^{q} |L_r|^{a_r - \frac{1}{2}(p+1)} |I - \sum_{r=1}^{q} L_r|^{b - \frac{1}{2}(p+1)}$. It follows from Tan (1960) that the integral of ϕ w.r.t. $L_1, \cdots, L_q$ over the space $Z = \sum_{r=1}^{q} L_r$ is

$$\int_{Z=\sum_{r=1}^{q} L_r} \phi(L_1, \cdots, L_q) \prod_{r=1}^{q} dL_r = (\prod_{r=1}^{q} \Gamma_p(a_r)/\Gamma_p(a)) |Z|^{a - \frac{1}{2}(p+1)} |I - Z|^{b - \frac{1}{2}(p+1)}. \tag{2.4}$$

Hence (2.3) can be written as

$$\int_{0<Z<I} (\int_{\sum_{r=1}^{q} L_r = Z} \phi(L_1, \cdots, L_q) \prod_{r=1}^{q} dL_r) C_K(RZ)\, dZ = (\prod_{r=1}^{q} \Gamma_p(\tfrac{1}{2}m_r)/\Gamma_p(\tfrac{1}{2}m)) \int_{0<Z<I} |Z|^{\frac{1}{2}(m-p-1)} |I - Z|^{\frac{1}{2}(n-p-1)} C_K(RZ)\, dZ.$$

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