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CONTINUITY OF THE BAYES RISK[1]

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In the theory of statistical decision functions it is sometimes desired to prove that the Bayes risk as a function of the prior distribution is concave and continuous on its domain. The property of concavity is immediate and this implies continuity on the *interior* of the domain if the parameter space is finite. This has been used e.g. by Lehmann [3], Lemma 3.12.5 and by Ferguson [2], Lemma 7.6.1. Continuity on the whole domain does not follow immediately. A proof applicable to a rather special sequential problem (finite parameter space, i.i.d. observations and constant cost per observation) has been given by Blackwell and Girshick [1], Theorem 9.4.2. Another continuity theorem, valid under certain restrictive conditions, can be found in Wald [4], Theorem 4.6. It is desirable to find conditions implying continuity that are both simpler and more widely applicable. The hope is to make strong use of the concavity of the Bayes risk. It should be realized, however, that even on a convex Euclidean domain a concave function is not necessarily continuous on the boundary. On the other hand, it will be shown in this note that continuity is implied by the combination of concavity and a very simple property of the geometry of the domain. This property is satisfied, for instance, by a polyhedron in finite dimensional space. Thus, it can be concluded that in any statistical problem with finite parameter space (plus a mild assumption on the risk functions) the Bayes risk is continuous.

In any given statistical problem let Θ denote the parameter space, whose points θ index the distributions on the sample space; and let D be any class of decision functions δ. We assume that to each $\delta \in D$ there corresponds a *risk function* R_δ on Θ satisfying $0 \leqq R_\delta(\theta) < \infty$ for all $\theta \in \Theta$, $\delta \in D$ (the uniform lower bound 0 could be replaced by any other). We further assume that there is given a sigma-field B on Θ such that every R_δ is B-measurable. Let Λ be a class of probability distributions λ on (Θ, B). The only requirement on Λ is that it be convex (e.g. Λ could be all probability distributions). Let $r_\delta(\lambda) = \int R_\delta(\theta)\, \lambda(d\theta)$ be the average risk of δ when the prior λ is used. Clearly, r_δ is linear on Λ. Define the *Bayes risk* $\rho(\lambda) = \inf_{\delta \in D} r_\delta(\lambda)$. ρ is concave on Λ since it is the infimum of a family of concave functions.

Suppose Λ carries a topology in which all r_δ are upper semicontinuous (often it will be possible to assert that the r_δ are continuous by virtue of their linearity). Then ρ, being the infimum of the r_δ, is also upper semicontinuous on Λ. We would also like to be able to conclude lower semicontinuity of ρ, but this seems impossible without further assumptions. Indeed, it is easy to give an example of a concave function f on a convex set in the plane that is not lower semicontinuous: let f equal 1 on the interior of a disk and in one point x of the boundary, and 0 elsewhere on

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