The Annals of Mathematical Statistics
1970, Vol. 41, No. 3, 1068–1070

CONVERGENCE OF SUMS TO A CONVOLUTION OF STABLE LAWS

BY J. DAVID MASON

University of Georgia

1. Statement of the problem. Let $\{X_n\}$ be a sequence of independent random variables such that the distribution function of X_n is one of $F_1, \cdots, F_r$, where $F_1, \cdots, F_r$ are r distribution functions and $F_i \in \mathscr{D}(\lambda_i)$, $i = 1, \cdots, r$. Here, $\mathscr{D}(\lambda)$ denotes the domain of attraction of the stable type with characteristic exponent λ. Assume $0 < \lambda_1 < \cdots < \lambda_r \leqq 2$. Let $n_i(n)$ be the number of random variables among $X_1, \cdots, X_n$ which have F_i as their distribution function, $i = 1, \cdots, r$. Assume $n_i(n) \to \infty$ as $n \to \infty$, for each i.

If we assume that there are constants $\{A_n\}$ and $\{B_n\}$, with $0 < B_n \to \infty$ as $n \to \infty$, such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a nondegenerate distribution function G, then Theorem 2 of [1] gives a necessary and sufficient condition (without the assumption that $F_i \in \mathscr{D}(\lambda_i)$) that G be a convolution of r distinct stable laws. The purpose of this paper is to obtain a necessary and sufficient conditon (with the assumption that $F_i \in \mathscr{D}(\lambda_i)$ but without the assumption of the existence of $\{A_n\}$ and $\{B_n\}$) that G be a convolution of $l \leqq r$ distinct stable laws.

2. Statement of theorem. Let $X(i, m)$ be the mth random variable in the sequence $\{X_n\}$ whose distribution function is F_i, $i = 1, \cdots, r$. Then, for each i, there are constants $\{A(i, n)\}$ and $\{B(i, n)\}$, with $0 < B(i, n) \to \infty$ as $n \to \infty$, such that $(B(i, n))^{-1}(X(i, 1) + \cdots + X(i, n)) - A(i, n)$ converges in law to a stable distribution with characteristic exponent λ_i. By Lemma 5 of [2], for each i, there exists a measurable slowly varying function L_i defined over $(0, \infty)$ such that $B(i, n) \sim n^{\lambda_i^{-1}} L_i(n)$. By Karamata's representation theorem,

$$L_i(x) = c_i(x) \exp\{\textstyle\int_0^x (\theta_i(t)/t)\, dt\},$$

where $c_i(\cdot)$ is a measurable function such that $c_i(x) > 0$ for all x and $c_i(x) \to c_i > 0$ as $x \to \infty$, and $\theta_i(\cdot)$ is Lebesgue-integrable over every finite interval $(0, x)$ and $\theta_i(t) \to 0$ as $t \to \infty$.

THEOREM. *A necessary and sufficient condition that there exist constants $\{A_n\}$ and $\{B_n\}$, with $0 < B_n \to \infty$ as $n \to \infty$, such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a nondegenerate distribution function G is that for some set of indices $\{i_1, \cdots, i_l\}$, with $1 \leqq i_1 < \cdots < i_l \leqq r$,*

$$B(j, n_j(n))/B(i_l, n_{i_l}(n)) \to p_j \qquad as \quad n \to \infty,$$

where $p_j = 0$ for $j \notin \{i_1, \cdots, i_l\}$ and $0 < p_j < \infty$ for $j \in \{i_1, \cdots, i_l\}$.

Furthermore, G is a convolution of l stable laws with characteristic exponents $\lambda_{i_1}, \cdots, \lambda_{i_l}$, and there is a constant $b > 0$ such that $B_n = bB(i_l, n_{i_l}(n))$ for all n.

Received July 14, 1969.