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WEAK CONVERGENCE OF PROBABILITY MEASURES ON THE FUNCTION SPACE $C[0, \infty)$[1]

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1. The space $C[0, \infty)$. Let $C \equiv C[0, \infty)$ be the set of all continuous functions on $[0, \infty)$ with values in a complete separable metric space (E, m). Stone (1961, 1963) has obtained simple criteria for weak convergence of sequences of probability measures on $\mathscr{C}$, the σ-field generated by the open subsets of C, when C is endowed with the topology of uniform convergence on compacta, cf. [4] page 229. We shall obtain further properties of $(C, \mathscr{C})$ by defining a metric ρ on C which induces this same topology.

For any two functions x and y in C, let $\rho : C \times C \to R$ be defined as

$$\rho(x, y) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(x, y)/[1 + \rho_j(x, y)],$$

where $\rho_j(x, y) = \sup_{0 \leqq t \leqq j} m[x(t), y(t)]$.

THEOREM 1. *The function space (C, ρ) is a complete separable metric space in which* $\lim_{n\to\infty} \rho(x_n, x) = 0$ *if and only if* $\lim_{n\to\infty} \rho_j(x_n, x) = 0$ *for all* $j \geqq 1$.

COROLLARY 1. *The metric topology in (C, ρ) is the topology of uniform convergence on compacta.*

Since the proofs of Theorem 1 and Corollary 1 are straightforward, we omit them.

Let $\mathscr{M}_p(C)$ be the set of all probability measures on $\mathscr{C}$. A net of probability measures $\{P_\alpha\}$ in $\mathscr{M}_p(C)$ is said to converge weakly to a probability measure P in $\mathscr{M}_p(C)$ if

$$\lim_\alpha \int_C f\, dP_\alpha = \int_C f\, dP$$

for every bounded continuous real-valued function f on C, and we write $P_\alpha \Rightarrow P$. Since (C, ρ) is a complete separable metric space, cf. [5] II. 6,

COROLLARY 2. *The space $\mathscr{M}_p(C)$ with the topology of weak convergence is metrizable as a complete separable metric space.*

The metric defined by Prohorov (1956) is one such metric, cf. [1] page 237.

We now wish to characterize the σ-field $\mathscr{C}$. For each $t \geqq 0$, let $\pi_t : C \to E$ be the coordinate projection, defined for any $x \in C$ by $\pi_t(x) = x(t)$. Let E be a measurable space with the σ-field generated by the open subsets and let E^k be the k-fold product of E with itself endowed with the product topology and the corresponding σ-field

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