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TRANSLATING GAUSSIAN PROCESSES[1]

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0. Introduction. Let μ_x denote the measure on path space corresponding to a stochastic process X. A function m is an admissible translate of X if μ_x and μ_{x+m} (alternatively $d\mu_x(\omega)$ and $d\mu_x(\omega - m)$) are mutually absolutely continuous. The problems of determining conditions for equivalence and of finding the corresponding Radon–Nikodym derivative $d\mu_{x+m}/d\mu_x$ (alternatively the Jacobian of the path space transformation $\omega \to \omega - m$) have been widely studied for Gaussian processes.

For the stationary case, Parzen [1] showed that m is admissible if and only if for t in the parameter set of the process $m(t)$ can be written as $\int e^{i\lambda t} g(\lambda)\, dF(\lambda)$ for some g in $L^2(dF)$, where F is the spectral measure of the process. For the Wiener process Segal [2] showed that m is admissible if and only if $m(t)$ can be written as $\int_0^t g(s)\, ds$ for g in L^2 and for t in the parameter set. Completely general conditions for admissibility of translations of arbitrary Gaussian processes are now known. One form of these conditions is that $m(t)$ must be representable as $E(X_t \psi)$, where X_t is the random variable evaluation at time t and where ψ is some element in the Hilbert space spanned by X_t, t in the parameter set. Another form states that if $R(s, t)$ is the covariance of X, then m must be in the reproducing kernel Hilbert space with kernel R. Finally, if R is assumed continuous, the condition is that m must be in the range of $\mathbf{R}^{\frac{1}{2}}$ acting on $L^2[T]$, where $\mathbf{R}^{\frac{1}{2}}$ is the square root of the integral operator with kernel R, and where T is the parameter interval.

These conditions have been derived by varied methods. To clarify the relation of these results it is worthwhile, without going into the derivations, to show that the different forms of description do indeed describe the same set of translations. This is done in Section 1.

These descriptions have only an indirect probabilistic appeal. In particular, none gives a direct relation between what might be called the innovation structure of the process and the properties of its admissible translations. In Section 2 a new form of admissibility condition is given involving the innovations of the process X. In Section 3 the structure of the Radon–Nikodym derivative considered as a stochastic process is exposed.

Section 1. Let X be a mean zero Gaussian process with continuous covariance $R(s, t)$ over a real interval T. Denote by H_T the Hilbert space spanned by the functions $X_t(\omega)$, $t \in T$. Denote by $\mathbf{X}$ the integral operator from H_T to $L^2(T)$ with kernel $X(t, \omega)$ $(= X_t(\omega))$, which we may assume to be measurable.

PROPOSITION 1.
The following sets are the same:
1. *The range of* $\mathbf{R}^{\frac{1}{2}}$ *acting on* $L^2(T)$.

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