The Annals of Mathematical Statistics
1970, Vol. 41, No. 3, 865–875

FLUCTUATIONS WHEN $E(|X_1|) = \infty$ [1]

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1. Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables. Set $S_n = \sum_{k=1}^n X_k$ and $S_0 \equiv 0$. We define the two random variables, $N(\omega)$ and $M(\omega)$, by $N(\omega) = \sum_{n=0}^\infty I_{\{\omega: S_n(\omega) \geqq 0\}}$ and $M(\omega) = \sup_{n \geqq 0}\{S_n(\omega)\}$.

It is the purpose of this paper to study the problem of the finiteness of N, M, $E(N)$, and $E(M)$ in the case when $E(|X_1|) = \infty$. We obtain results which parallel known results for the case when $E(X_1)$ is finite and negative. In [7] it is shown that if $O > E(X_1) > -\infty$ and if k is a positive integer that $E(M^k) < \infty$ if and only if $E((X_1{}^+)^{k+1}) < \infty$. This difference of unity between the order of these moments appears as the difference between α and $\alpha + 1$ in our results. If $0 > E(X_1) > -\infty$ then it can be deduced from [3] and [6] and a truncation argument that $E((X_1{}^+)^2) < \infty$ if and only if $E(N) = \sum_{n=0}^\infty P(S_n \geqq 0) < \infty$. The fact that $E(N) < \infty$ implies $E((X_1{}^+)^2) < \infty$ when $E(X_1)$ is finite and negative, can also be obtained from Theorem 7 of [8] by setting $l = 2$. This ratio of 1 to 2 in the order of these moments appears as the ratio of α to 2α in our results and suggests conjectures concerning the existence of higher moments of N and $X_1{}^+$. In all that follows, $F(x) = P(X_1 \leqq x)$. All slowly varying functions, L, are assumed to have been defined so that $L(x) = L(-x)$.

PROPOSITION 1. *Assume there exists* $x_0 < 0$, *a constant* α *satisfying* $0 < \alpha < 1$, *and a function L slowly varying at* ∞, *such that for all* $x \leqq x_0$, $L(x)/|x|^\alpha$ *is monotone and* $F(x) \geqq L(x)/|x|^\alpha$. *Then*:

(i) $E((X_1{}^+)^\alpha / L(X_1{}^+)) < \infty$ *implies both N and M finite* a.s.;
(ii) If $\alpha \neq \frac{1}{2}$, $E((X_1{}^+)^{2\alpha}/L^2(X_1{}^+)) < \infty$ *implies* $E(N) < \infty$;
(iii) $E((X_1{}^+)^{1+\alpha}/L(X_1{}^+)) < \infty$ *implies* $E(M) < \infty$.

PROPOSITION 2. *Assume there exists* $x_0 < 0$, *a constant* α *satisfying* $0 < \alpha < 1$, *and a function L slowly varying at* ∞, *such that for all* $x \leqq x_0$, $L(x)/|x|^\alpha$ *is monotone and* $F(x) \leqq L(x)/|x|^\alpha$. *Then*:

(i) *Either N or M finite* a.s. (*hence both*) *implies* $E((X_1{}^+)^\alpha/L(X_1{}^+)) < \infty$;
(ii) $E(N) < \infty$ *implies* $E((X_1{}^+)^{2\alpha}/L^2(X_1{}^+)) < \infty$;
(iii) $E(M) < \infty$ *implies* $E((X_1{}^+)^{\alpha+1}/L(X_1{}^+)) < \infty$.

REMARK 1. If F and G are probability distribution functions with $F \leqq G$ for all x then by induction $F^{n*} \leqq G^{n*}$ for all x and all n. $L(x)/|x|^\alpha$ which is assumed monotone for $x \leqq x_0 < 0$ can be pieced together with F to form a probability

Received May 5, 1969.
[1] Research supported in part by NSF grant GP8890.