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A NOTE ON UNIFORM CONVERGENCE OF STOCHASTIC PROCESSES[1]

By Naresh C. Jain and G. Kallianpur

University of Minnesota

0. Introduction. Our aim in this note is to extend Theorems 5.1 and 5.2 of [4]. Let $R(\cdot, \cdot)$ be the covariance kernel of a Gaussian process with index set S, here S will always mean a compact metric space. R is assumed throughout to be continous on $S \times S$. Let $H(R)$ be the reproducing kernel Hilbert space of R. It is a Hilbert space of continuous functions k on S with the following properties:

$$R(\cdot, t) \in H(R) \qquad \text{for each} \quad t \in S; \tag{0.1}$$

$$\langle k, R(\cdot, t)\rangle = k(t), \tag{0.2}$$

where $\langle\, ,\rangle$ denotes the inner product in $H(R)$. For a discussion of reproducing kernel Hilbert spaces and their application to the study of Gaussian processes we refer to [1] and [5]. In what follows $C(S)$ will denote the Banach space of real-valued continuous functions on S with the sup norm, and $\mathscr{C}$ the σ-algebra of Borel sets of $C(S)$. x will denote a generic element of $C(S)$.

Before stating the main results we would like to record here for later reference the fact that if $\{X_t, t \in S\}$ is a Gaussian process on some probability space $(\Omega, \mathscr{F}, P)$, then there is an isometric isomorphism between $H(R)$ and the closure of the linear space spanned by $\{X_t, t \in S\}$ in $L_2(\Omega, \mathscr{F}, P)$. We shall denote this closure by $\mathscr{L}_2(X_t, t \in S)$ and this isometric isomorphism by θ, where for $t \in S$ we have $\theta(R(\cdot, t)) = X_t$. We now state the main results.

Theorem 1. *Let $\{X_t, t \in S\}$ be a Gaussian process with covariance R and almost all paths continuous on a complete probability space $(\Omega, \mathscr{F}, P)$. Let $\{\psi_j\}_{j=1}^{\infty}$ be a complete orthonormal system* (CONS) *in $H(R)$ and let $\{\xi_j\}_{j=1}^{\infty}$ be the sequence of independent random variables on $(\Omega, \mathscr{F}, P)$ each distributed normally with mean 0 and variance 1, given by $\xi_j = \theta(\psi_j)$. Then the partial sums*

$$\textstyle\sum_{j=1}^{n} \xi_j(\omega)\psi_j(t) = S_n(t, \omega) \tag{0.3}$$

converge uniformly in $t \in S$ to $X_t(\omega)$ as $n \to \infty$ a.e. (P).

Theorem 2. *Let $\{\eta_j\}_{j=1}^{\infty}$ be a sequence of independent random variables on a complete probability space $(\Omega, \mathscr{F}, P)$, each distributed normally with mean 0 and variance 1. Let R be a covariance such that there exists a Gaussian process with this covariance and with almost all sample paths continuous (on some probability space). Let $\{\psi_j\}_{j=1}^{\infty}$ be a* CONS *in $H(R)$. If $S = [0, 1]$, then the partial sums*

$$\textstyle\sum_{j=1}^{n} \eta_j(\omega)\psi_j(t) = S_n'(t, \omega) \tag{0.4}$$

converge uniformly in $t \in [0, 1]$ to a Gaussian process on $(\Omega, \mathscr{F}, P)$ *whose covariance is R and almost all of whose sample paths are continuous as $n \to \infty$* a.e. (P).

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