The Annals of Mathematical Statistics
1970, Vol. 41, No. 4, 1357–1359

A SIMPLE PROOF OF AN INEQUALITY FOR MULTIVARIATE NORMAL PROBABILITIES OF RECTANGLES[1]

BY KUMAR JOGDEO

University of Illinois

Suppose $X_1, \cdots, X_k$ are jointly normally distributed random variables with zero means and correlation matrix $R = \{\rho_{ij}\}$. Then intuitively, it is clear that for a fixed set of positive constants, $c_1, \cdots, c_k$, the probability $P[|X_i| \leqq c_i, i = 1, \cdots, k]$ should increase when the correlation coefficients increase in some fashion. A precise result of this nature was obtained by Šidák (1968): if $R(\lambda)$ denotes the correlation matrix with $\rho_{1j}(\lambda) = \rho_{j1}(\lambda) = \lambda\rho_{1j}$ for $j > 1, 0 \leqq \lambda \leqq 1$; $\rho_{ij}(\lambda) = \rho_{ij}$ for all other i, j and P_λ denotes the probability measure corresponding to $R(\lambda)$ then

$$\frac{d}{d\lambda} P_\lambda[|X_i| \leqq c_i] \geqq 0. \tag{1}$$

This result has several interesting applications and as pointed out by Šidák it presents a partial analog to a result by Slepian (1962), obtained in connection with "one sided barrier" problem:

$$\frac{d}{d\rho_{ij}} P[X_l \leqq c_l, l = 1, \cdots, k] \geqq 0 \tag{2}$$

where $c_1, \cdots, c_k$ are arbitrary real numbers. Slepian (1962) showed that (2) is an immediate consequence of the following equation. (Chartres (1963) gave a geometrical proof of (2)). Assuming (without loss of generality, as far as the statements of theorems in the present paper are concerned) that variances of X_i are unity, it may be verified that

$$\frac{d}{d\rho_{ij}} g(\mathbf{x}, R) = \frac{\partial^2}{\partial x_i \partial x_j} g(\mathbf{x}, R) \tag{3}$$

where g is the multivariate normal density of $X_1, \cdots, X_k$.

Unfortunately, Šidák's proof of (1) is very lengthy and from his remarks it seems as if Slepian's method is not workable for this "two sided" version. In the following it will be shown that Slepian's method when applied to a lemma given in this paper, readily gives the desired inequality. This lemma is a simple corollary to an inequality of Anderson (1955) which also served as the key to Šidák's proof.

As observed by Šidák it suffices to prove (1) under the assumption that R is nonsingular, since for the general case one may obtain a sequence of nonsingular matrices approaching the given one and the inequality would still be preserved. Henceforth, R will be assumed to be nonsingular.

Received April 7, 1969.

[1] This research supported by NSF Grant GP 14786.