The Annals of Mathematical Statistics
1970, Vol. 41, No 4, 1354–1356

A CHARACTERIZATION OF CERTAIN INFINITELY DIVISIBLE LAWS

BY ALAN RUEGG

University of Connecticut

1. In the theory of infinitely divisible (i.d.) distribution functions (df's), it is well known that a finite df (i.e. a df whose entire mass is concentrated on a finite interval) cannot be i.d. unless it is degenerate. Different proofs of this result have been given, most of them in connection with the investigation of one-sided df's (see [1], [3], [6], [7]). The purpose of the present note is to generalize the above statement, i.e. the following question will be answered: How "close" can a non-degenerate i.d. df F be to a finite df, or more precisely: How rapidly can the "tail" T of F, given by $T(x) = 1-F(x)+F(-x)$, converge to zero as $x \to \infty$ if F is a non-degenerate i.d. df ?

2. THEOREM 1. *If F is* i.d., *and if there exist constants $a > 0$ and $\alpha > 1$ such that $T(x) = 0[\exp(-ax^{1+\alpha})]$ as $x \to \infty$, then F is degenerate.*

If F is finite, the above hypothesis holds for any positive α; Theorem 1 therefore generalizes the result mentioned in **1**.

THEOREM 2. *If F is* i.d., *non-degenerate, and if there exist constants $a > 0$ and $\alpha(0 < \alpha \leqq 1)$ such that $T(x) = 0[\exp(-ax^{1+\alpha})]$ as $x \to \infty$, then F is normal.*

PROOF OF THEOREMS 1 AND 2. By Theorem 7.2.4. ([4] page 142), the characteristic function (ch.f.) f of F is an entire function of finite order $\rho_f \leqq 1+\alpha^{-1}$. Since F is i.d., f has no zeros ([4] page 187), and therefore $f(z) = \exp(g(z))$, where g denotes the principal determination of $\log f$, vanishing at $z = 0$.

By the definition of ρ_f, we have for every positive ε

$$\begin{aligned}\max_{|z|=r} \mathscr{R} g(z) &= \max_{|z|=r} \log|f(z)| \\ &= \log \max_{|z|=r} |f(z)| \leqq r^{\rho_f + \varepsilon}\end{aligned}$$

for all sufficiently large r, hence by Theorem 1.3.4. ([2] page 3), g is a polynomial and its degree is equal to ρ_f. But a classical result due to Marcinkiewicz ([4] page 147) states that the only ch.f.'s which have the form $\exp(g(z))$, g being a polynomial, are either $\exp(-az^2+ibz)$ (normal law) or $\exp(ibz)$ (degenerate law) with respective orders of 2, 1 or 0, and since $\rho_f \leqq 1+\alpha^{-1}$, the assertions of Theorems 1 and 2 follow immediately.

COROLLARY 1. *The only* i.d. ch.f.'*s which are entire functions of finite order are the normal and the degenerate* ch.f.

3. By using a different and slightly more involved method of proof, the hypothesis of Theorem 2 can be weakened in the following way.

Received December 1, 1969.