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CONVERGENCE PROPERTIES OF S_n UNDER MOMENT RESTRICTIONS[1]

BY R. J. SERFLING

Florida State University

1. Introduction and preliminaries. Let $\{X_i\}_{-\infty}^{\infty}$ be a sequence of rv's having finite variances $\{\sigma_i^2\}$. Assume throughout (without loss of generality) that $E(X_i) \equiv 0$. For each vector $\mathbf{X}_{a,n} = (X_{a+1}, \cdots, X_{a+n})$ of n consecutive X_i's, let $F_{a,n}$ denote the joint df and let

$$S_{a,n} = \sum_{a+1}^{a+n} X_i. \tag{1.0}$$

In statements about $\mathbf{X}_{0,n}$ only, the abbreviated notation $\mathbf{X}_n$, F_n, S_n, etc., shall be employed.

This paper concerns stochastic convergence properties of sequences $\{S_{a,n}\}_{n=1}^{\infty}$. It will suffice to prove statements about the sequence $\{S_n\}_{n=1}^{\infty}$, but the more general notation will be of use in formulating some of the restrictions adopted.

Convergence properties of the following types shall be discussed:

$$P[S_n/n \to 0] = 1; \tag{1.1}$$

$$\sum_1^{\infty} a_n P[\sup_{k \geqq n} |S_k/k| > \varepsilon] \quad \text{converges for every } \varepsilon > 0; \tag{1.2}$$

$$P[\limsup_{n\to\infty} |S_n/b_n| \leqq 1] = 1; \tag{1.3}$$

$$\sum_1^{\infty} c_n P[\sup_{k \geqq n} |S_k/b_k| > \varepsilon] \quad \text{converges for every } \varepsilon > 0; \tag{1.4}$$

$$\sum_1^{\infty} P[|S_n/d_n| > \varepsilon] \quad \text{converges for every } \varepsilon > 0. \tag{1.5}$$

In the above, $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ and $\{d_i\}$ are sequences of constants.

Condition (1.1) expresses the strong law of large numbers (SLLN) for the sequence $\{X_i\}_1^{\infty}$ and condition (1.2) represents information regarding the rate of the convergence in (1.1). The larger that the a_n's may be chosen (in asymptotic order of magnitude), the sharper is the result stated by (1.2). Condition (1.3) expresses a form of the law of the iterated logarithm (LIL) (e.g., in typical cases b_n may be taken as small as $O((n \ln \ln n)^{\frac{1}{2}})$.) and condition (1.4), like (1.2), concerns the rate of convergence. Finally, condition (1.5) states that the sequence $\{S_n/d_n\}_{n=1}^{\infty}$ *converges completely* to zero in the sense of Hsu and Robbins [7]. The smaller that the d_n's may be chosen, the sharper is the statement. By the Borel–Cantelli lemma, complete convergence implies strong convergence.

Properties such as (1.1)–(1.5) will be obtained as consequences of restrictions imposed upon the absolute νth moments, for some $\nu \geqq 2$, of sums $\sum_{a+1}^{a+n} w_i X_i$, where the w_i are given constants (e.g., $w_i \equiv 1$, or $w_i = \ln i$). *Thus it is not assumed that the X_i's are mutually independent and, in fact, the only restrictions on the dependence will be what is implied by the restrictions of the type mentioned.* (See [12], [13] for details.)

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